Periodic signals with minimal power fluctuations

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In this article, Pumplin's algorithm [J. Pumplin, J. Acoust. Soc. Am. **78**, 100–104 (1985)] is used to find periodic waveforms with minimal power fluctuations. Starting with a particular power spectrum, the algorithm can find a set of phases for the harmonics such that the variance in waveform power is a minimum or near a minimum. Such optimized waveforms are smooth and tend to have very small crest factors. The particular power spectra chosen for study include narrow- and wideband spectra, with emphasis given to signals that are useful in research in psychoacoustics and in physiological acoustics.

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INTRODUCTION

A given periodic waveform, such as a simple acoustical signal, has a unique power spectrum consisting of a set of harmonic levels. A given power spectrum, however, does not correspond to a unique periodic waveform. There are an infinite number of different periodic waveforms, corresponding to different phases of the harmonics, all with the same power spectrum. It frequently happens in communications systems that the information is primarily conveyed by the power spectrum. In such cases one is at liberty to choose the phase angles of the harmonics at will. Some choices of phase angles lead to spiky waveforms, where the signal energy is transmitted in sharp pulses. Other choices of phase angles lead to smooth waveforms. Mathematically, the differences between these waveforms can be described in terms of power fluctuations. Spiky waveforms have large power fluctuations, smooth waveforms have small power fluctuations.

In a practical communications system there is an advantage to minimizing the power fluctuation. Waveforms with minimum fluctuation transfer the greatest amount of power with a small dynamic range in waveform. Usually, this leads to communications with the greatest signal-to-noise ratio and the least distortion. In the scientific study of communications systems, one frequently wants to control the amount of power fluctuation as a parameter. For instance, waveforms with the same power spectrum but with different power fluctuations can be used in psychoacoustical studies of monaural phase sensitivity. (Buunen, 1976; Patterson, 1987) or in corresponding physiological studies (e.g., Deng *et al.*, 1987). In general, one expects that the effects of nonlinear distortion in a system, such as the human auditory system, are most easily seen the greater the difference in power fluctuation.

Fluctuations can be measured in terms of crest factor, defined as the ratio of the maximum absolute value of the waveform to the rms value. For a given power spectrum, it is easy to generate the waveform that has the largest possible crest factor: One simply adds up cosines with the correct amplitudes and with the phases equal to zero. The maximum value of the waveform occurs at time zero, and it is equal to the sum of the amplitudes (all positive numbers). This is as large as a waveform with the given spectrum can possibly be.

An alternative measure of fluctuation, called the "peak factor" (PF) by Schroeder (1970), is defined as half the difference between the waveform maximum and the waveform minimum, divided by the rms value. For a waveform that is symmetrical about zero, the peak factor is identical to the crest factor. Schroeder also defined the "relative peak factor" (R_{peak}) as relative to the peak factor for a sine waveform. Therefore, R_{peak} is equal to $\text{PF}/\sqrt{2}$. As in the case of the crest factor, the relative peak factor is maximized by a choice of phase angles that are all zero, where each harmonic is written as a cosine. For the efficient transmission of information, this choice of phases is the worst of all.

More interesting than the worst choice of phases, is the attempt to find the best, the choice that leads to the smallest fluctuation. Beyond its practical implications, a minimum fluctuation is an ideal, but, apart from a few simple cases, the search for such optimum waveforms has proved to be difficult. There has always been a vast territory to explore, and although one might find something that appears to be the minimum-fluctuation waveform, it was always possible that a waveform with still smaller fluctuation might be waiting on the other side of the next hill.

Schroeder's 1970 paper presented a simple algorithm for choosing the phases of the harmonics of a waveform in order to obtain a small peak factor. The virtues of this algorithm are that it is exceedingly easy to use and that it can be applied in general for power spectra that are smooth and dense. When the algorithm works, it works rather well; the fluctuations in the waveforms produced by the algorithm are usually small.

There are, however, signals of interest where Schroeder's algorithm cannot be applied. For example, if the power spectrum consists only of a first harmonic (fundamental) and a third harmonic, the algorithm actually leads to the worst possible choice of phases, a choice that maximizes the peak factor.[1] A further difficulty with waveforms generated by Schroeder's algorithm is that they are chirps, predominantly one frequency followed in time by another. If the system of interest has high Q and small minimum integration time, e.g., the human auditory system, then the algorithm is restricted to cases where the spacing between the harmonics is large compared to the inverse of the integration time.

An alternative measure of fluctuations is the variance of the power, computed over the period of the signal. Minimizing the variance of power is equivalent to minimizing the fourth moment of the waveform.[2] In 1985, Pumplin developed an algorithm to minimize moments. This algorithm is completely general. It works for any power spectrum, and the waveforms that it produces seem to have no peculiar features apart from the fact that they have very small fluctuations.

Experience gained in applying this algorithm (Hartmann and Pumplin, 1988) led to the realization that those waveforms, created to minimize the fourth moment, also have small crest factors. It was then conjectured that the choice of phase angles that minimizes the fourth moment also leads to small values of the moments of arbitrary order. If this is true, then waveforms with a minimized fourth moment have small fluctuation by any measure of fluctuation that one might choose.

Our experience also found no instance in which Pumplin's algorithm fails. Unfortunately, however, the algorithm does not give a closed-form expression that can be used in the general case. One must apply the algorithm's search technique anew to each power spectrum of interest.

The purpose of the present article is to provide useful minimum-fluctuation waveforms. We chose power spectra that have been used, or are likely to be of use in experimental work in psychoacoustics and in physiological acoustics. We then applied Pumplin's algorithm to these spectra in a straight-forward way. Except in a few cases, the resulting minimum-fluctuation waveform could be described only by a table of the phase angles for the harmonics. Some of our tables appear below; more extensive tables are in a document called Minimum Power Fluctuations Extended Tables (MPFET), available from the Physics Auxiliary Publication Service.[3]

I. THE PROBLEM AND THE PROCEDURE

A. Definitions

A periodic waveform as a function of time, with period T, is given by a sum of harmonics,

$$x(t) = \sum_{n=N_1}^{N_2} A_n \cos\left(\frac{2\pi nt}{T} + \phi_n\right), \tag{1}$$

where N_1 and N_2 are the minimum and maximum harmonic numbers, and the total number of harmonics is $N = N_2 - N_1 + 1$. In the tables below, the phase angles ϕ_n will be reported in units of radians, between 0 and 2π.

The fourth moment that we minimize is the quantity W

$$W = \overline{x^4}/(\overline{x^2})^2, \tag{2}$$

where the bar indicates a time average over one period. Here, W is normalized by the square of the waveform power, which appears in the denominator. The waveform power is independent of the choice of phase angles, and therefore the normalization plays no role in the actual minimization of the fourth moment.

B. The procedure

The procedure for minimizing the fourth moment operates in the multidimensional space of phase angles. Each harmonic has a phase that is a variable in the minimization procedure. The dimension of the space is equal to the number of harmonics less one, because one phase can be chosen arbitrarily without changing the shape of the waveform. This space is the playing field where minimization attempts take place. One can move around in it looking for optimum waveforms. Pumplin's algorithm provides a directed search, based upon a gradient search procedure. Beginning at an arbitrary starting point in the space, the algorithm finds a path where the fourth moment of the waveform descends abruptly into a local minimum. Crucial to the application of the algorithm is the fact that all the partial derivatives that comprise the gradient in the multidimensional space can be determined rapidly using a single FFT. After falling into a local minimum, a hole in the multidimensional space, the gradient search stops.

In general, we find that the space of phase angles is full of holes, discrete and narrow and distributed in the space with no apparent regularity. Some holes are deeper than others, and the object of our procedure is to fall into the deepest hole. Therefore, we choose a succession of different starting points at random, falling into a nearby hole each time. In the end, we remember the coordinates of the deepest local minimum, and this set of phase angles is our final answer. With enough random starts we expect to find the global minimum, but the algorithm cannot guarantee that the deepest minimum that is found in a finite number of starts is actually the global minimum.

We gained some insight into the statistics of the solutions from the algorithm by studying cases in which the amplitudes of the harmonics are all equal. This condition leads to the most rugged and challenging terrain of all the spectra in this article. Wandering around in this space, falling into millions of different holes, we have learned some things.

(1) The number of holes in the space Z grows rapidly as the number of harmonics in the spectrum increases. The growth is approximately exponential. For waveforms with $N_1 = 1$, the function $Z \approx \exp[0.51(N_2 - 5.5)]$ describes the growth of the number of minima in the space when the top harmonic number N_2 becomes large. When the number of harmonics is not large (e.g., 12 so that $Z \approx 28$, it frequently happens that different starting points lead into the same hole. In these cases, only a few starts (e.g., 1000) are necessary to be rather confident that the best local minimum that one finds is actually the global minimum. When the number of harmonics exceeds 24 ($Z \approx 12500$) there are so many holes in the space that we need about 1 000 000 starts to be confident of finding the deepest hole in the space.

(2) For a given spectrum, the great majority of holes have similar depths. The different local minima correspond to fourth moments that are usually within a range of several percent. For this reason, it does not matter much whether we find the global minimum or not. Most holes are approximately as good as the best hole.

(3) The fourth moment at any local minimum is considerably smaller than the fourth moment at an arbitrarily cho-

sen point in the space: A factor of 2 is typical.

(4) The local minima are narrow. When the number of harmonics is ten or more, there is virtually no chance of landing near the bottom of a hole by random jumps that do not follow a path down into the hole. For this reason, it is extremely unlikely that the popular experimental choice of random-phase signals (or noise) would ever result accidentally in a minimum-fluctuation waveform, or anything approaching it.

(5) Except in simple cases, the local minimum with the smallest fourth moment is not the local minimum with the smallest crest factor. However, by choosing holes with the smallest fourth moment, we obtain crest factors that are only a few percent larger than the smallest crest factor. Generally, our recent experience has confirmed the previous idea that minimizing the fourth moment leads to quite small crest factors.

The above points influenced our method of calculating optimum waveforms. For each new power spectrum we began at a number of different starting points. When the number of harmonics was small, it was sufficient to make only 1000 starts, doing the calculations on a VAX 8650. For the larger numbers of harmonics, we used as many as 1 000 000 different starting points, on a Convex 240. Using such a large number of starting points on a supercomputer was not done in order to find a lowest fourth moment of practical importance. As noted in (2) above the different holes have very similar depths. Rather, our search for the absolute deepest hole was motivated by the hope that a pattern would emerge from the global minima, a pattern that would guide further analytic work. In the end, some patterns did emerge, as described below, though these patterns had less generality than one might have hoped for. As a result of the extensive calculations, we are able to present global minima in the present article and for those waveforms in MPFET with less than 30 components.

C. Widebands and narrow bands

The bandwidth is determined by the minimum harmonic number N_1 and the maximum harmonic number N_2. There are tables below for two situations, widebands and narrow bands. We define a band to be "narrow" if its center frequency, $(N_2 + N_1)/2$, is greater than the bandwidth $N_2 - N_1$ (Hartmann and Pumplin, 1988). An equivalent definition of a narrow band is that N_2 is smaller than three times N_1. This definition is not obvious; most people would probably not refer to a band that extends from 50 to 149 Hz as narrow. However, this definition of narrow band is correct in the present application because it defines the case in which the fourth moment is independent of the center frequency. Such independence leads to a generalization of considerable power because it is possible to translate a minimum-fourth-moment calculation rigidly along the frequency axis (linear Hz), i.e., to add a constant integer to all harmonic numbers.

II. WIDEBAND SIGNALS

Most of the signals of ordinary speech and music are wide band. Idealized as steady-state tones, they have a fundamental component and a series of harmonics. For a wideband signal, the phase angles that minimize the fluctuations in the waveform depend upon the number of harmonics and upon the relative levels of the harmonics. We examine here several such signals.

A. Flat spectra

1. Results

A common choice in experimental work is to let all the harmonics, up to some maximum harmonic number, have the same level. Our results for spectra of this type are shown in Table I. The entries there show the phase angles, the set $\{\phi\}$, that minimize the fourth moment for eight different spectra, arranged by columns. In the first column are the phases that minimize the power fluctuation when the spectrum consists of the first three harmonics, all of equal amplitude. In the second column are the phases when the spectrum consists of the first four harmonics, etc. Therefore, from Eq. (1), an optimized signal with fundamental frequency of 100 Hz and four harmonics is given by the equation

$$x(t) = \cos(2\pi 100t) + \cos(2\pi 200t + 4.875) + \cos(2\pi 300t + 5.363) + \cos(2\pi 400t + 1.465). \quad (3)$$

In each column of Table I, the phases are followed by a check sum. This is simply a sum of all phase angles; it can be used as a check on data entry. Next in the column is the fourth moment, the quantity W. For comparison, the fourth moment for a single sine wave is 3/2. The relative variance in power can be obtained by subtracting 1 from the value given for the fourth moment. Because the relative variance must be non-negative, the fourth moment cannot be less than unity.

TABLE I. Wideband equal amplitude. A column of the table shows the phases that are needed to make a waveform that has a minimum fourth moment by adding harmonics according to Eq. (1), where the lowest harmonic number is 1 and the highest may take on values from 3–10. For example, the column $N = 5$ shows the phases to be used when the spectrum consists of the first five harmonics. This table applies to waveforms where all nonzero amplitudes are the same.

N	3	4	5	6	7	8	9	10
Harmonic	Phases (radians)							
1	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
2	0.000	4.875	$3\pi/2$	5.436	5.170	5.399	5.238	5.079
3	π	5.363	π	3.373	0.242	4.729	5.201	4.921
4		1.465	$3\pi/2$	5.032	5.576	2.021	3.689	3.650
5			0.000	0.679	2.349	5.544	0.734	0.400
6				6.143	3.009	0.643	5.132	5.554
7					0.177	1.056	0.300	0.775
8						3.174	1.505	2.062
9							3.583	4.077
10								0.980
Check	π	11.703	4π	20.664	16.523	22.566	25.382	27.496
Fourth	1.611	1.503	1.420	1.579	1.404	1.444	1.387	1.381
Crest	1.659	1.523	1.607	1.671	1.460	1.707	1.537	1.505
R_{peak}	1.002	1.060	1.136	1.123	0.998	1.152	1.057	1.053

The standard deviation in power, relative to the power itself, can be obtained by taking the square root of the relative variance. For instance, the standard deviation in power for a sine wave is the square root of $(3/2 - 1) = 0.707$. The final entries in the table are the crest factor and the relative peak factor.

Tables IA and IB (both in MPFET[3]) are extensions of Table I to larger numbers of harmonics, all of equal amplitude. Table IA gives phases for N_1 equal to 1 and N_2 taking on increasing values from 11 through 18; Table IB presents phases for N_1 equal to 1 and N_2 equal to 20, 24, 31, 36, 48, and 60.

Figure 1 puts the minimized power fluctuations into perspective. The values of the fourth moment from the waveforms of Tables I and IA are shown by closed circles as a function of the number of harmonics. These values are the global minima for each value of N. The values from Schroeder's algorithm are shown by symbols S in the figure. It is clear that Schroeder's algorithm never achieves the absolute minimum value, though it does remarkably well, considering its simplicity.

The plot labeled with open symbols O shows the fourth moment when the harmonics are all in "sine phase" ($\phi_n = 3\pi/2$ for all n). This function continues to rise with increasing N, linearly with a slope of 2/3 in the limit that N becomes infinite. (See the Appendix for a proof of this.) It is clear that this popular choice of phases leads to a waveform that has relatively enormous power fluctuations. If the number of harmonics is greater than 4, any set of phase angles chosen at random has an excellent chance of achieving smaller power fluctuations than "sine phase."

The remaining data in Fig. 1 show the results of a calculation, where 10 000 different waveforms were generated by picking the phases randomly from a uniform distribution. Such choices correspond to the starting points for our algorithm, without following any path. The average fourth moment of these random starts is given by squares, the median (50th percentile on the distribution) is given by triangles. The mean is always larger than the median because the distribution has a long tail toward larger fourth moments (Hartmann, 1987). Lines show values for 10th and 90th percentiles, e.g., 90% of the randomly generated waveforms have fourth moments that are less than the 90th-percentile value.

The line labeled "min" shows the smallest value ever seen in 10 000 waveforms. It shows that when there were only six harmonics in the waveform, we could do almost as well by jumping around randomly in the space as by following the algorithm, provided that we took the best of 10 000 jumps. However, when there were seven or more harmonics, 10 000 jumps was not enough to approach the value obtained using the algorithm.

Figure 2 gives a further view of fluctuations. The closed circles show the crest factor for the waveforms of Tables I and IA. The closed squares show the average crest factor, averaged over 1000 waveforms with randomly chosen phases. The open circles show the crest factor for waveforms constructed with sine phase. Again, it is evident that sine-phase waveforms have large fluctuation compared to the average random-phase waveforms, and the difference increases with increasing number of harmonics. The waveforms from Pumplin's algorithm have a crest factor that is about 60% of the average crest factor for random waveforms. Still smaller values of the crest factor can be obtained by minimizing higher moments.

2. Phase freedom

There are three distinct operations that can be performed on a set of phase angles that leave the moments of the waveform unchanged. First, one can add a constant value of π to all the phase angles. This simply has the effect of inverting the waveform [$x(t)$ goes to $-x(t)$]. Second, one can change the signs of all the phases, which corresponds to reversing the direction of time [$x(t)$ goes to $x(-t)$]. Third,

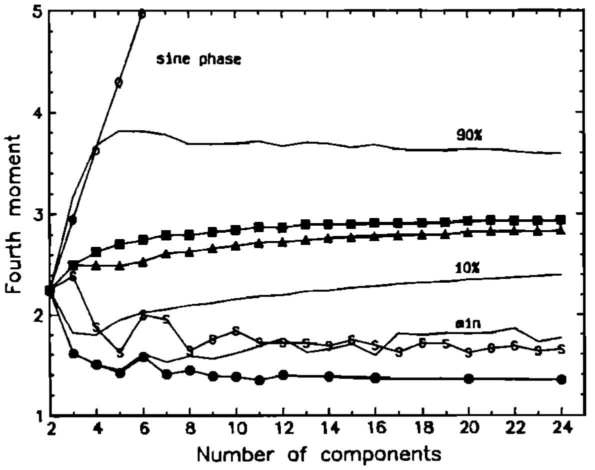

FIG. 1. Fourth moments are shown for wideband equal-amplitude spectra, having all harmonics up to N, where N is given on the horizontal axis. The phases have been chosen in four different ways: Pumplin's algorithm (closed circles), Schroeder's algorithm (symbols S), sine phase (symbols O), and random choice with mean (closed squares) and median (closed triangles). For the random choice, the 10th and 90th percentiles, and the minimum value for 10 000 random waveforms are given by solid lines.

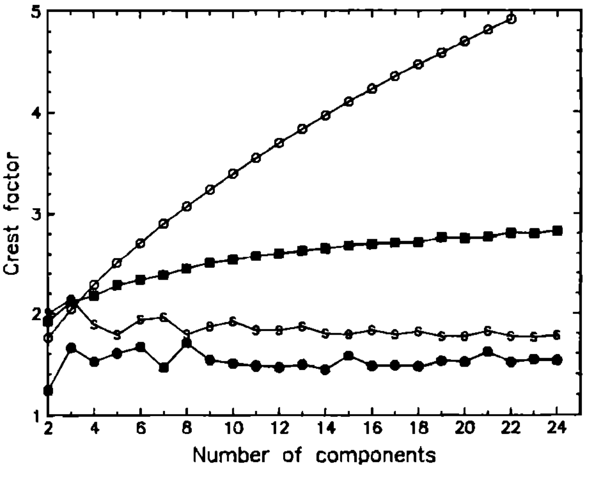

FIG. 2. Crest factors are shown for wideband equal-amplitude spectra, having all harmonics up to N, where N is given on the horizontal axis. The phases have been chosen in four different ways: Pumplin's algorithm (closed circles), Schroeder's algorithm (symbols S), sine phase (symbols O), and random choice with the mean shown by closed squares.

one can add a phase shift to each harmonic that is proportional to the frequency of that harmonic. This operation is equivalent to choosing a different origin for time.[4] These freedoms apply to any waveform, regardless of number of components or amplitudes. They could be used to make some changes in the waveforms from our tables. For example, in the case of the three-harmonic waveform, the optimum phases were found to be $\{0,0,\pi\}$ corresponding to a series of trigonometric functions, cos, cos, $-$cos. Adding $n\pi/2$ to each phase, where n is the harmonic number (freedom 3), gives phase angles $\{\pi/2,\pi,\pi/2\}$. Then, subtracting π from each phase (freedom 1) gives the phase angles $\{-\pi/2,0,-\pi/2\}$, corresponding to the series sin, cos, sin. The last waveform is equivalent to the first.

B. Octave spectra

Tones in which all the harmonics are octaves of the fundamental can be used to study pitch chroma while minimizing the pitch-height information. (See, for example, Ueda and Ohgushi, 1987). Band-passed versions of such signals are the so-called Shepard tones (Shepard, 1964). Tones with octave spectra have the highly unusual property that the choice of phases that minimizes the fourth moment is independent of the amplitudes of the harmonics! This fact is proved in the Appendix. This means that once one has created a minimum-fluctuation version of the waveform, subsequent bandpass filtering does not change its minimum-fluctuation character, so long as the filtering introduces no phase distortion. Further, an optimal choice of phases is easy to describe: It is simply to let all phase angles be $-\pi/2$, making a series of sine components. This was actually Shepard's original choice. In the Appendix it is shown that there is an additional freedom in connection with these octave spectra. For example, a series of cosine components with alternating sign gives the same fourth moment.

Although the series of sines and the series of alternating-sign cosines minimize the fourth moment regardless of the component amplitudes, the actual value of the fourth moment, and of the other fluctuation statistics too, depend upon the amplitudes. These values are given in MPFET[3] for the case that all octave harmonics have the same amplitude, for overall ranges of 2, 3, 4, 5, or 6 octaves.

C. Other equal-amplitude spectra

Recently, Plomp (1989) demonstrated some equal-amplitude spectra with interesting implications for studies of pitch and timbre. The phases for minimum fluctuation are given in Table II. The first two columns are for bell tones that have been "regularized" by forcing inharmonic bell spectra to be harmonic. The column labeled "minor" is a choice of harmonic numbers to simulate a traditional bell, the column labeled "major" is a choice of harmonics that resembles the new bells based upon a major chord (Lehr, 1987).

The third column of Table II is for a tone that resembles a pipe organ sound, probably because of its strong octave cue. Successive harmonics are separated by more than 1/3 octave so that no more than one harmonic lies in any critical band.

TABLE II. Equal-amplitude signals: pitch and timbre studies. The first three columns give phases that minimize the fourth moment for spectra, idealized from musical instrument tones, used in timbre studies. The four columns to the right give phase angles for spectra with missing fundamentals. For those harmonics where no phase is given the amplitude is zero. Otherwise all amplitudes are the same.

	Bells major	Bells minor	Pipe organ				
Harmonic	Phases (radians)			$N=5$	$N=6$	$N=7$	$N=7$
1			0.000				
2	0.000		4.988	0	0	0	
3			0.585	$7\pi/4$	π	0	0
4	4.881		1.126	$3\pi/2$	$2\pi/3$	0	5.958
5	3.164	0.000		$\pi/4$	$2\pi/3$	π	1.368
6	3.452		4.921	π	π	π	0.887
7					0	0	4.888
8	4.322		3.538			π	5.960
9							3.142
10	0.003	1.733					
12		5.548	5.548				
15		5.363					
16			4.303				
20		4.606					
24		2.760	2.759				
Check	15.822	20.010	27.768	$9\pi/2$	$10\pi/3$	3π	22.203
Fourth	1.787	2.251	1.691	1.660	1.694	1.520	1.590
Crest	1.961	2.128	1.803	1.698	1.719	1.556	1.725
R_{peak}	1.222	1.482	1.247	1.201	1.215	1.074	1.214

The remaining columns, to the right in Table II, show phases that minimize fluctuations for periodic tones with missing fundamentals. A large number of missing-fundamental tones can be created using the narrow-band results to be described below. However, a few important cases do not qualify as narrow band and these are the ones that are given in Table II. These, together with the cases given for narrow bands, are an adequate base for equal-amplitude missing-fundamental experiments, such as those of Singh (1987), Tomlinson and Schwartz (1988), or Zatorre (1988).

D. Bright and dark tone colors

A signal with many harmonics of equal amplitude has a bright (or buzzy) tone color. More natural sounding tones have harmonics that decrease in amplitude with increasing harmonic number. Amplitudes that decrease as the inverse qth power of the harmonic number form a common idealization. If amplitudes decrease as the inverse first power of the harmonic number ($q=1$), the spectrum decreases at a rate of approximately -6 dB/oct. Such a spectrum is still rather bright in comparison with the sounds of musical instruments and speech. The latter are better represented by $q=2$ or $q=3$, approximately -12 and -18 dB/oct, respectively. Table III shows phase angles that lead to minimum- fluctuation waveforms for q equal to 1, 2, or 3.

For q equal to 2 or 3, there is a quite general optimization rule. The optimum phase angles are given by the se-

TABLE III. Wideband spectra with uniformly-decreasing amplitudes. Phases that minimize the fourth moment for amplitude spectra having harmonics that decrease as the inverse cube (− 18), the inverse square (− 12), or the inverse first power (− 6) of the harmonic number. Where no phase is given for a harmonic the amplitude is zero.

N	5	5	10	16	5	10	20
dB/oct.	− 18	− 12	− 12	− 12	− 6	− 6	− 6
Harmonic	Phases (radians)						
1	0	0	0	0	0.000	0.000	0.000
2	0	0	0	0	5.581	5.750	5.954
3	π	π	π	π	2.228	2.393	2.652
4	π	π	π	π	3.213	3.504	3.566
5	0	0	0	0	6.198	0.677	1.187
6			0	0		4.482	5.189
7			π	π		6.095	0.327
8			π	π		2.594	3.282
9			0	0		5.224	1.133
10			0	0		1.318	3.135
11				π			5.723
12				π			2.573
13				0			5.150
14				0			1.883
15				π			4.589
16				π			0.269
17							3.891
18							0.812
19							2.450
20							5.174
Check	2π	2π	4π	8π	17.220	32.037	58.941
Fourth	1.464	1.407	1.393	1.393	1.376	1.262	1.203
Crest	1.515	1.519	1.537	1.520	1.354	1.345	1.349
R_{peak}	0.963	0.894	0.884	0.879	0.945	0.916	0.909

quence $0,0,\pi,\pi,0,0,\pi,\pi,\ldots$ for successive harmonics, beginning with the first. This corresponds to the series: cos, cos, − cos, − cos,... . According to the phase freedom rules, this sequence is equivalent to the sequence of alternating trigonometric functions: sin, cos, sin, cos... . To find the optimum waveform with a finite number of harmonics, one simply truncates the sequence as needed. This procedure works for any number of harmonics greater than 3. It also works for nonintegral values of q greater than some value, approximately equal to 2. This optimization rule is a generalization of our experience;[5] we have not yet been able to prove it.

The alternating cosine-sine series, which works so well for q equal to 2 or greater, fails badly for the case of a flat spectrum. For example, for an equal-amplitude spectrum with the first ten harmonics, the alternating cosine-sine series gives a fourth moment of 4.450 and a crest factor of 2.802. These compare unfavorably with the averages for random-phase waveforms given in Figs. 1 and 2. For a larger number of equal-amplitude harmonics, the series with alternating phases compares even more unfavorably. As in the case of constant phase, the fourth moment of a signal with alternating phases increases linearly with the number of components. This point is discussed further in the Appendix.

III. NARROW-BAND SIGNALS

A signal is classified as narrow band if the full bandwidth (highest frequency minus lowest frequency) is less than the center frequency of the band. In the case of narrow-band signals, the fourth moment of the waveform is equal to 3/2 times the fourth moment of the envelope, as shown in Appendix B of Hartmann and Pumplin (1988). Therefore, the envelope phase rules (deBoer, 1961) apply to the waveform fourth moment: (1) One can shift the phase of each harmonic by an amount that is proportional to the harmonic frequency (simply a time shift), and (2) one can shift all the phases by a constant amount. Both kinds of shift leave the fourth moment invariant. For wideband signals, only the first invariance holds. Phases that lead to minimum power fluctuations for narrow-band signals having three to ten equal-amplitude harmonics are given in Table IV.

The fact that the fourth moment is tied to the envelope fourth moment means that the case of narrow bands differs in two rather important respects from the case of wide bands. First, a narrow-band solution that minimizes the fourth moment can be translated rigidly along the frequency axis and still maintain its minimizing character. For example, Table IV shows the solution that minimizes the fourth moment for a waveform consisting of harmonics 3,4,5,6,7. Because of the generality of narrow-band signals, the user can legitimately translate this solution along the frequency axis to any other central component greater than 4, for example, to the case of harmonics 99, 100, 101, 102, and 103. Both cases are narrow band; the minimizing phase solution for one case is a minimizing phase solution for the other. The crest factors for these two cases will, however, be different. One expects that the crest factor will be larger for the signal centered at harmonic 101, but it will not be much larger.

Second, for narrow-band signals, there are two phase angles, instead of one, that are arbitrary so far as the fourth moment is concerned. The time-translation freedom accounts for one of these, and the other arises from the fact that the fourth moment is not changed by adding a constant to all the phases (see the Appendix). Therefore, instead of finding

TABLE IV. Narrow-band equal-amplitude signals. Phases that minimize the fourth moment for narrow-band signals with equal-amplitude harmonics, for numbers of harmonics from 3–10.

N	3	4	5	6	7	8	9	10
Harmonic	Phases (radians)							
2	0.000	0.000						
3	5.498	5.523	0.000	0.000				
4	1.571	2.939	5.760	5.837	0.000	0.000		
5		4.815	5.236	1.621	6.200	6.010	0.000	0.000
6			1.571	6.202	1.850	4.583	6.054	6.078
7			4.189	0.729	1.611	5.240	0.436	4.740
8				4.053	5.474	0.396	0.002	4.064
9					0.882	2.618	3.732	0.047
10					4.590	0.640	2.552	6.077
11						3.792	5.537	2.235
12							1.140	6.029
13							3.919	2.411
14								4.277
Check	7.069	13.277	16.755	18.441	20.608	23.279	23.372	35.959
Fourth	1.833	1.781	1.740	1.917	1.631	1.629	1.539	1.635

isolated holes in the space of phase angles, the algorithm finds trenches. As one moves along the bottom of a trench, the fourth moment is unchanged, and therefore, a procedure that simply minimizes the fourth moment is inadequate to find a single set of optimum phase angles. The features of narrow-band signals are discussed further in the Appendix.

We used this extra degree of freedom to advantage by obtaining a small sixth moment. Starting at a random location in space, we used the gradient of the fourth moment to fall into a trench. Then, we walked along the bottom of the trench until we found the point of smallest sixth moment. This optimized calculation was used to find the phase angles for equal amplitudes and narrow bands given in Table IV, and in two extended tables in MPFET,[3] Tables IVA and IVB. In Table IVA, the number of harmonics goes from $N = 11$ through 17. Table IVB gives phases for $N = 21$, 24, 31, 36, and 48 harmonics. For each value of the number of harmonics in the spectrum, N, the tables show the solution for the smallest harmonic numbers that satisfy the narrow-band requirement. However, as noted above, the solutions can be translated along the frequency axis to find other waveforms with a minimized fourth moment. The waveform that results from this translation will not, however, have the minimized sixth moment. The minimized sixth is peculiar to the particular band of harmonics used for our calculation.

When the frequency range of the narrow-band signal is translated, the sixth moment and higher moments are not expected to become very large. The fact that the fourth moment is minimized tends to restrain the higher moments. However, higher moments, such as the crest factor may become large enough that it is possible to find lower values. For example, Preece and Wilson (1988) found a crest factor as low as 1.84 for a waveform with five harmonics, numbers 51–55. Translating the waveform from Table IV with $N = 5$ leads to a crest factor of 1.90. The waveform from Table IV, of course, has a smaller fourth moment, 1.74 vs 1.85.

IV. SIMPLE WAVESHAPES—DISTORTED

Simple wideband waveforms with geometrical shapes that are easy to describe are created by electronic function generators; the square wave and the triangle wave are examples. This section gives phase angles that minimize the power fluctuations, as measured by the fourth moment, for signals having the power spectra of these simple waveforms. Although the shapes of these optimized waveforms are quite different from their geometrically simple ancestors (we call them pseudowaveforms below), informal listening tests indicate that the optimized waveforms sound quite similar to their ancestors, as would be expected from Ohm's phase law. Perceptible consequences of the optimization are subtle.

A. The pseudotriangle wave

The power spectrum of the triangle wave consists of all odd harmonics, with levels decreasing at a rate of -12 dB/oct. The phase spectrum can be described most simply by saying that the triangle is the sum of harmonics that are sine functions with alternating signs. The fourth moment is 1.8, the crest factor is $\sqrt{3}(=1.732)$, and the relative peak factor is $\sqrt{3/2}(=1.225)$. Numerical calculations with the algorithm found that the waveform that has the same power spectrum and minimum fluctuation is simply the sum of cosine functions with alternating signs; i.e., it is identical to the triangle except that the cosine function always replaces the sine function. The fourth moment is $W = 1.2956$, the crest factor is 1.2861, and the relative peak factor is 0.90942. A plot of this wave, together with a triangle wave having the same power, is shown in Fig. 3.

B. The pseudo-half-wave

The half-wave-rectified sine has no odd harmonics except for the first. It can be represented by the Fourier series,

$$x = \frac{1}{\pi} + \frac{1}{2}\sin(\omega t) - \frac{2}{\pi}\sum_{n=2,4,6,\ldots}^{\infty} \frac{1}{n^2-1}\cos(n\omega t), \tag{4}$$

where $\omega = 2\pi/T$.

Figure 4 was made by keeping terms $n = 2,4,\ldots,24$ in the sum. Because the dc term ($n = 0$, having a value of $1/\pi$) has been omitted from the figure, the half-wave has zero average value, unlike a true half-wave rectified sine. The fourth moment of the half-wave with zero average is 1.741, the crest factor is 1.767, and the relative peak factor is 0.9205.

Numerical calculations with the algorithm to minimize the fourth moment find that the optimized waveform is given by the series[6]

$$x(t) = \frac{1}{2}\sin(\omega t) - \frac{2}{\pi}\sum_{n=2,4,6\ldots}^{\infty} \frac{1}{n^2-1}\sin(n\omega t). \tag{5}$$

The result is the peculiarly slanted waveform shown in Fig. 4 along with the half-wave. Its fourth moment is 1.729, the crest factor is 1.517, and the relative peak factor is 1.072. The relative peak factor is actually higher than that for the undistorted half-wave.

C. The pseudopulse waveform

1. The simple pulse

A pulse waveform has two possible values, high and low. An example is the square wave, where the duty factor is $p = 1/2$. For other values of the duty factor, p ($p < 1$), there is finite average value in the waveform. If this dc component is removed then the fourth moment is given by

$$W = 1/[p(1-p)] - 3. \tag{6}$$

The crest factor is given by

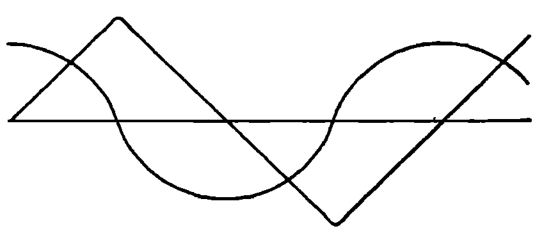

FIG. 3. The triangle wave and the pseudotriangle. Both waveforms have the same power spectrum, consisting of harmonics 1,3,...,23. The pseudotriangle has the smallest possible power variance.

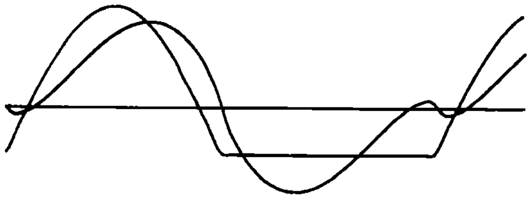

FIG. 4. The half-wave and the pseudo-half-wave. Both waveforms have the same power spectrum, consisting of harmonics 1,2,4,...,24. The pseudo-half-wave waveform has the smallest possible power variance.

$$C=\sqrt{(1-p)/p},\quad \text{for } p\leqslant 0.5,$$
$$C=\sqrt{p/(1-p)},\quad \text{for } p>0.5, \tag{7}$$

and the relative peak factor is

$$R_{\text{peak}}=1/\sqrt{8p(1-p)}. \tag{8}$$

The pulse wave can be written as a Fourier series, a sum over harmonics, n, of functions that are all cosines, having amplitudes given by

$$A_n=(1/n)\sin(np\pi). \tag{9}$$

By convention, we let the amplitude for each harmonic be given by the absolute value of Eq. (9) and let the phase be either 0 or 180 deg depending upon whether the sine function in Eq. (9) is positive or negative.

If the duty factor p is the reciprocal of an integer, then harmonics that are integral multiples of that integer have zero amplitude. For example, if $p = 1/3$, then the third, sixth, ninth,... harmonics are missing from the power spectrum.[7]

2. Application of the algorithm

For every pulse waveform, with a given duty factor, there is a pseudopulse with a minimum fourth moment. To find the pseudopulses, we proceeded, as for other pseudowaveforms, by starting with the power spectrum, here given by Eq. (9), for a finite number of harmonics. We use Pumplin's algorithm to find the phase angles corresponding to the minimum-fluctuation waveform. Our study concentrated on pseudo-pulses where the parent pulse waveform had a duty factor of the form 1/integer. The application of the algorithm led to a number of surprises described below.

First, it is evident that the fourth moment and the crest factor of the square wave $p = 1/2$ are already as low as possible; both are equal to one. What also appears in our calculations, though it is not *a priori* evident, is that the fourth moment of a pulse with $p = 1/3$ is also as low as possible, $W= 3/2$.

For other values of the duty factor, of the form $p = 1/$integer, it is not hard to find pseudopulses with fourth moments that are less than the fourth moment of the corresponding pulse. Table V gives the phase angles that minimize the fourth moment for $p = 1/4$, and 1/7, when the pulse is represented by 60 Fourier components. Blanks appear in the table where phases cannot matter because the amplitude is zero. The table includes the fourth moments of the waveforms, and also (in parentheses) the fourth moments for the undistorted original pulses. The comparison shows that minimization becomes increasingly important as p decreases, as would be expected: Pulse waveforms are the most spiky when p is small. Tables for $p = 1/5$, 1/6, 1/8, and 1/10 appear in MPFET.[3]

Table V does not include crest factors or relative peak factors. It is not clear how to deal with these measures of fluctuation for a pulse that is represented by a finite number of Fourier components. Attempting to represent the discontinuity in the pulse waveform by a Fourier series with a finite number of terms leads to Gibbs phenomena, which are damped oscillations at the points of discontinuity. These oscillations distort the calculated values of extrema that determine the crest factor and relative peak factor.

Figure 5 shows one period of the pseudopulse, with minimized fourth moment, computed from the entries in Table V for $p = 1/4$. This waveform is typical of one class of pseudopulses. For this class, there is no hint of the original duty factor. The function is extremely ragged, as though the jolts that occur at the two discontinuities of the original pulse are spread out over the entire period. The minimized solution is stable with respect to the number of components. If, for example, the number of harmonics is increased to 256, then the first 60 phases are essentially the same as in Table V. The waveform is essentially the same too. The only change is that all parts of the waveform acquire small rapid oscillations representing the higher Fourier components.

3. City-scapes

A second class of minimum-fluctuation pseudopulse is the city-scape. We define a city-scape waveform as a series of rectangles of different heights, but all of the same width:

$$x(t)=C_0,\quad \text{for } 0\leqslant t<\tau,$$
$$x(t)=C_1,\quad \text{for } \tau\leqslant t<2\tau, \tag{10}$$
$$\cdots$$
$$x(t)=C_{M-1},\quad \text{for } (M-1)\tau\leqslant t<T,$$

TABLE V. Pseudo pulse. Phases for pseudopulse waveforms made from 60 harmonics of pulse waveforms with duty factors 1/4 and 1/7. Phases and check sums are in radians, entries are blank where the amplitude is zero. The fourth moment is given for the pseudopulse; the fourth moment for the pulse follows in parentheses. The table should be read from left to right, top to bottom.

Duty factor 1/4, Check = 135.940, Fourth = 1.2846 (2.3333)

0.000	6.217	3.014		3.008	5.823	1.349		2.631	4.509
1.002		1.036	4.328	3.021		0.993	0.098	3.644	
3.294	0.636	4.547		3.762	1.482	6.086		4.676	3.170
0.764		0.294	4.174	2.572		1.212	6.060	3.600	
3.340	0.960	5.053		4.473	2.324	6.060		6.060	3.402
0.616		0.942	4.442	1.591		2.259	5.338	2.077	

Duty factor 1/7, Check = 158.487, Fourth = 1.1094 (5.1667)

0.000	5.097	1.468	2.685	5.368	4.099		0.951	6.139	2.481
3.695	0.128	5.051		1.904	0.897	3.493	4.704	1.177	5.995
	2.862	1.939	4.505	5.711	2.235	0.646		3.826	2.981
5.517	0.430	3.305	1.568		4.799	4.026	0.243	1.427	4.394
2.467		5.786	5.078	1.249	2.414	5.518	3.318		0.512
6.146	2.245	3.381	0.428	4.046		1.580	0.996	3.253	4.326

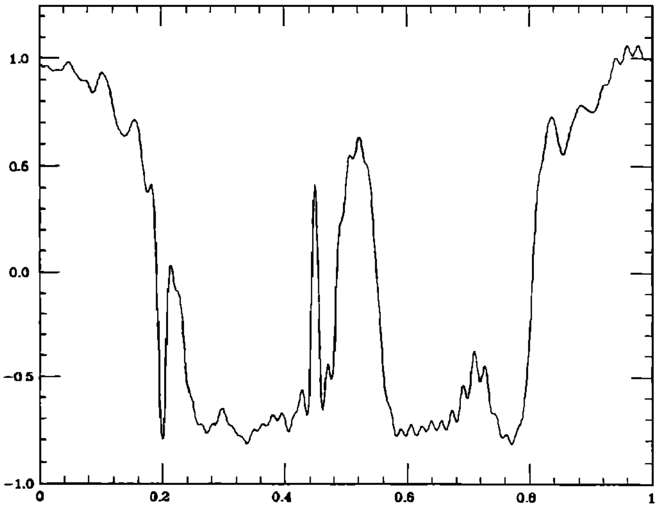

FIG. 5. The pseudopulse with 25% duty factor, computed from 60 components with phases given by Table V. The pseudopulse has the same energy as a pulse that has values $+1$ and -1, offset so that the average value is zero.

where τ is some integral fraction of the period; i.e., $\tau = T/M$. An example occurs for our pseudopulse with $p = 1/7$, as shown in Fig. 6. The solution corresponds to the series of constants $\{C_n\}$ equal to 3, 3, -4, -4, 3, -4, 3. Thus there are $M = 7$ buildings in the city, with zero average height. For this ideal waveform, the fourth moment is 1.083, the crest factor is 1.155, and the relative peak factor is 0.714. The waveform in Fig. 6 was computed with only 60 harmonics, and therefore the discontinuities in the function show Gibbs oscillations. There are even Gibbs oscillations between buildings in the city that have the same height, somehow representing the continued influence of the original duty factor of 1/7.

A city-scape waveform, with $M = 3$ buildings, is the

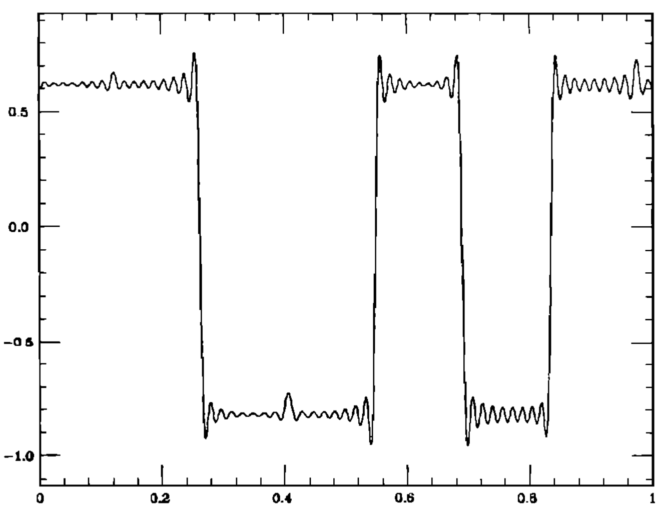

FIG. 6. The pseudopulse with a duty factor of 1/7, computed from 60 components with phases given by Table V. The ideal waveform has only two possible values, but Gibbs phenomena are everywhere here. There is no fragmentation. The pseudopulse has the same energy as a pulse that has values $+1$ and -1, offset so that the average value is zero.

minimum-fluctuation solution for the case $p = 1/3$. This case has the peculiar property that any choice of the three constants, so long as they add up to zero, gives the same spectrum as the $p = 1/3$ pulse (which corresponds to $C_1 = C_2 = -C_0/2$). Further, any choice of the constants gives the same fourth moment ($W = 1.5$).

The phases in MPFET[3] for $p = 1/6$ also correspond to a city-scape, as shown in the figure in MPFET.[3] That waveform shows the phenomenon of "city-scape fragmentation." Instead of having $M = 6$ buildings, the waveform has 12. The fragmentation factor is 2.

The fragmentation factor must be an integer. This can be proved simply by looking at the zeros of the power spectrum. A pulse with duty factor of $p = 1/L$ has a power spectrum with zero power at harmonic numbers $L, 2L, 3L, \ldots$. It is not hard to show that a city-scape with M buildings has a power spectrum with zeros at harmonic numbers $M, 2M, 3M, \ldots$. Therefore, the spectral zeros of pulse and city-scape are the same if $M = L$.

Suppose now that one wants to make a city-scape with twice as many buildings. The spectral zeros needed to make this waveform are at harmonic numbers $2M, 4M, 6M, \ldots$. All of these zeros are actually present in the spectrum for the pulse with duty factor $p = 1/L$, where $L = M$. The same argument holds good if one wants to multiply the number of buildings by any integer. The spectrum of the pulse allows one to fragment the city scape in this way. If however, one wants to multiply the number of buildings by a factor that is not an integer, then the set of zeros that are available in the spectrum for a pulse with duty factor $p = 1/L$ ($L = M$) is not adequate to provide all the zeros that one needs to make the city-scape. In particular one cannot even find the lowest-frequency zero.

The minimum-fluctuations solution given in the MPFET[3] for $p = 1/10$ is also a city-scape. In this case, the fragmentation factor is 4. The possibility of very large fragmentation factors raises an interesting problem in the study of city-scapes. At some point the fragmentation may become so extreme that one cannot recognize the difference between a city-scape and some other ragged waveform. Or possibly a city-scape might be recognized as such only for a large number of Fourier components in the spectrum.

The discovery of the low-fluctuation city-scape waveform raises other questions that are beyond our present analytical and computational capabilities. We know that city-scapes frequently arise as low-fluctuation solutions. For example, although our solution in MPFET[3] for $p = 1/5$ is not a city-scape, we have discovered a waveform with a fourth moment that is almost as small that *is* a city-scape. We cannot predict when the optimum solution will be a city-scape and when it will not.

D. The pseudosawtooth

In their classic study of monaural phase perception Plomp and Steeneken (1969) used a spectrum with the first ten harmonics with harmonic levels decreasing at -6 dB/oct. If the waveform is made by summing sine waves (phase of 270 deg in our notation), then the resulting waveform is a sawtooth, and discarding all but the first ten harmonics is equivalent to low-pass filtering a sawtooth waveform.

Plomp and Steeneken used four different sets of phase. They particularly emphasized that waveforms generated with alternating cosine and sine series (0 and 270 deg) sound different from the low-passed sawtooth. Here, we would simply note that the alternating cosine-sine series does not have the lowest power fluctuations (fourth moment or crest factor) for this spectrum. Minimum power fluctuations occur for the phase angles given in Table III with $N = 10$ and a spectral envelope decreasing at -6 dB/oct. The waveform is shown in Fig. 7. It may be compared with waveforms given by Plomp and Steeneken in their Fig. 1. Of their four waveforms, number 3 has the smallest fourth moment and crest factor, respectively, 1.507 and 1.708.

V. CONCLUSION

Using Pumplin's algorithm for minimizing moments of a waveform, we have found optimized periodic waveforms, having minimum power fluctuation, for a variety of power spectra. Minimum fluctuation waveforms were found by choosing the phases of the harmonics to minimize the fourth moment. We considered both wideband and narrow-band spectra.

Among the wideband signals were flat spectra and spectra decreasing at -6, -12, and -18 dB per octave. Also included were octave spectra (as used in making Shepard tones), regularized bells and pipes, and the pseudowaveforms corresponding to triangle, half-wave, pulses, and sawtooth. For signals with a spectrum that decreases at -12 dB/oct or faster, we found that the optimum choice of phases does not depend upon the number of harmonics included in the waveform. This fortunate result considerably increases the general usefulness of our calculations. When the power spectrum decreases at -6 dB/oct, the optimization does depend upon the number of harmonics that are included in the waveform, but the dependence is not sensitive. An experimenter who wants an optimized waveform with this power spectrum would do well simply by truncating a column from Table III.

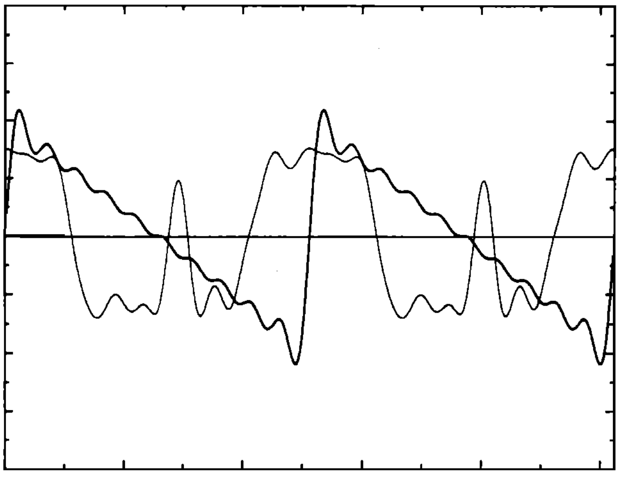

FIG. 7. The sawtooth (heavy) and the pseudosawtooth (light). Two cycles of the waveforms are shown. Both waveforms have the same power and the same power spectrum, consisting of harmonics $n = 1 - 10$, decreasing in amplitude as $1/n$, as studied by Plomp and Steeneken in 1969.

For narrow-band spectra special considerations apply. To obtain a single optimized solution, we considered the continuum of solutions that minimize the fourth moment and selected the one with the smallest sixth moment. This choice also led to small values of higher moments such as the eighth. The optimized solution has the convenient property that it can be translated rigidly along the frequency axis while maintaining the fourth moment at a minimum. In other words, the solution for seven adjacent partials in a narrow band at a low frequency is also the correct solution for seven adjacent partials at a higher frequency. The solutions for narrow bands should be of particular importance in pitch and timbre experiments with missing low harmonics.

Because of its extremely robust nature, Pumplin's algorithm essentially finds *the* answer to the question about which waveform has the least power fluctuation. For a number of harmonics equal to 20 or less, we are rather sure that no one will ever find a waveform with a smaller average fluctuation.[8] In this sense then, we claim to have found ideal minimum-fluctuation waveforms. Of course, one may question our choice of criterion, whereby the fluctuation is measured by low-order moments such as the fourth.

The fourth moment is a direct measure of the variance of the power. Therefore, the fourth moment is the best overall measure of fluctuation in the same way that the standard deviation of a distribution is the best overall indicator of the width of the distribution. By minimizing the fourth moment, our calculations ensure that the fluctuation is best minimized in an average way. For certain applications, however, one might be more interested in a different measure of the fluctuation, such as the crest factor, in the same way that one might choose to characterize the width of a distribution by the range of its outliers.

There is, though, more to recommend the procedure of minimizing the fourth moment than simply the fact that the fourth moment is the best measure of fluctuation in the overall sense. Our calculations find that by minimizing the fourth moment, we frequently achieve extremely low values for higher moments such as the crest factor. In fact, we get lower values of the crest factor than are found with Schroeder's algorithm.

A final example will illustrate this point. Schroeder demonstrates his algorithm by calculating a waveform that has a spectrum with sixteen harmonics under a spectral envelope that is a squared sine function. His waveform has a fourth moment of 1.539, a crest factor of 1.756, and a relative peak factor of 1.17. These values are considerably smaller than are obtained with random phases: Schroeder's algorithm works well in this case. Applying our methods to this spectrum, we find a waveform with a fourth moment of 1.442, a crest factor of 1.575, and a relative peak factor of 1.10. It is no surprise that our fourth moment is lower; we have found the lowest one that there is. What is surprising is that our procedure, without particularly trying to do so, has led to smaller values of the crest factor and relative peak factor.

Although we have found the key to minimum-fluctuation waveforms, it is worth remembering that a minimum-fluctuation waveform may not really represent the ideal from the point of view of perception. Even with perfect electromechanical transducers, a waveform still undergoes phase distortion as it propagates on the basilar membrane. (Allen, 1983; Smith *et al.*, 1986). For a variety of perceptual and physiological experiments, especially for broadband signals, the ideal minimum-fluctuation waveform may be one that has been predistorted so as to compensate for the phase distortion of the cochlea. Such waveforms can be constructed based upon models of the cochlear filter. One begins with amplitude and phase spectra as given in our tables and multiplies by the transfer function of the inverse filter. The inverse Fourier transform then gives the optimized waveform within the context of the cochlear model.

ACKNOWLEDGMENTS

Dr. G. Allen kindly sent us power spectra for four vowels from Klatt's synthesis program [D. H. Klatt, J. Acoust. Soc. Am. 67, 971–995 (1980)]. The amplitudes and the phases that minimize the power fluctuation are given in the MPFET. This work was supported in part by the National Institute on Deafness and other Communication Disorders of the National Institutes of Health under Grant No. DC00181.

APPENDIX: ANALYTIC RESULTS FOR THE FOURTH MOMENT

The fourth moment is the best overall representation of the power fluctuations of a signal. The optimized waveforms in the body of this article were found by minimizing the fourth moment by numerical calculation, which appears to be the only way to answer most of the interesting questions about minimum-fluctuation waveforms. However, there are a few results that can be obtained analytically. These are treated in this Appendix.

We begin by deriving an expression that represents the fourth moment as a quadruple sum over the harmonics of the signal. From Eqs. (1) and (2) of the body, the *unnormalized* fourth moment is

$$\overline{x^4} = \frac{1}{T}\int_0^T dt \left(\sum_n A_n \cos(2\pi nt + \phi_n)\right)^4. \tag{A1}$$

Changing integration variables and expanding the fourth power of the sum, we find

$$\begin{aligned}\overline{x^4} = \sum_{n,j,k,l} A_n A_j A_k A_l \int_0^1 & dt \cos(2\pi nt + \phi_n) \\ & \times \cos(2\pi jt + \phi_j)\cos(2\pi kt + \phi_k) \\ & \times \cos(2\pi lt + \phi_l).\end{aligned} \tag{A2}$$

The product of four cosine functions can be written as a sum of single cosines. There are eight such single cosine terms. Because of the symmetry of the sum, the eight terms can be combined into three:

$$\begin{aligned}&\tfrac{1}{8}\cos(a+b+c+d) + \tfrac{3}{8}\cos(a+b-c-d) \\ &\quad + \tfrac{4}{8}\cos(a+b+c-d),\end{aligned} \tag{A3}$$

where a is the argument of the first cosine in (A2), namely $2\pi nt + \phi_n$, and b is the argument of the second, etc.

To obtain the fourth moment from Eq. (A2), the expression (A3) must be integrated. The simplification afforded by the form (A3) is that the integral of a single cosine function is zero unless the sum of the harmonic numbers in its argument is zero. For example, unless there is a dc component (zero frequency), the first term in (A3) can only integrate to zero because a sum of positive numbers can never equal zero. The other two terms in (A3) select out other terms in the sum on harmonic numbers.

The result of the integration is

$$\begin{aligned}\overline{x^4} = \tfrac{1}{8}\sum_{n,j,k,l} & A_n A_j A_k A_l \\ & \times [\delta_{n+j+k+l,0}\cos(\phi_n + \phi_j + \phi_k + \phi_l) \\ & + 3\delta_{n+j,k+l}\cos(\phi_n + \phi_j - \phi_k - \phi_l) \\ & + 4\delta_{n+j+k,l}\cos(\phi_n + \phi_j + \phi_k - \phi_l)].\end{aligned} \tag{A4}$$

Equation (A4) is the basic equation from which all subsequent results will be derived. There are three kinds of terms in the sum, with three different requirements on the harmonic number indices n, j, k, and l. The requirements are enforced by the Kronecker delta function, which is unity if the integer indices separated by the comma are the same, and is zero otherwise. The first terms in the square brackets are zero unless there is a dc component in the waveform. Only then can four harmonic numbers add up to zero. In what follows we shall assume that the dc component has been removed from the waveform. Therefore, the harmonic number indices are all positive integers. The second kind of terms are those where two pairs of indices must add in a way to be equal. We shall call these the "pairs" terms. Finally are the kind of terms where the sum of three indices must equal a fourth. We shall call these the "3-1" terms.

1. Two components

The general waveform with two sine harmonics is

$$f(x) = \cos(2\pi pt) + a\cos(2\pi qt + \phi), \tag{A5}$$

where p and q are integers with q greater than p, and a is any real number.

The "3-1" terms can contribute to the sum only if q is equal to three times p. If this is not the case, then only the "pairs" terms contribute. There are two terms where all indices are equal, and four where only the sums of pairs are equal. Therefore, the unnormalized fourth moment is $\frac{3}{8}(1 + a^4 + 4a^2)$, independent of phase ϕ. The fourth moment, normalized by the square of the average power, is given by

$$W = \tfrac{3}{2}(1 + 4a^2 + a^4)/(1 + a^2)^2. \tag{A6}$$

It does not matter which of the two harmonics is the larger, the expression for W is unchanged if a is replaced by $1/a$. The maximum value of W (equal to 9/4) occurs when $a = 1$. Because W is independent of phase ϕ, one cannot minimize W to determine an optimum value of ϕ, per Pumplin's algorithm.

In the exceptional case, q is equal to three times p. Then the contribution from the "pairs" terms are the same as before, but there is a single "3-1" term which introduces a phase dependence. The normalized fourth moment is then given by

$$W=[\tfrac{3}{2}(1+4a^2+a^4)+2a\cos(\phi)]/(1+a^2)^2. \tag{A7}$$

In this case Pumplin's algorithm does find the correct solution, namely $\phi=180$ deg. This solution also gives the lowest crest factor for the case of a 3 to 1 frequency ratio.

But, the case of two harmonics is not particularly interesting because the fluctuation is not strongly dependent on the phase angle. For instance, in a waveform with first and second harmonics of equal amplitude, the worst case occurs when the two harmonics are in phase ($\phi=0$) and the crest factor is 1.414. In the best case ($\phi=90$ deg) the crest factor is 1.245, only 12% smaller. Numerical experiments with other pairs of consecutive harmonics show that the improvement on optimization is even smaller than 12%.

2. Narrow bands

Equation (A4) gives some insight into the important case of narrow bands. For a narrow band, the highest frequency is less than three times the lowest frequency. Thus, there is no way that three harmonic numbers can add up to equal a fourth harmonic number and the "3-1" terms do not contribute. By definition, a narrow band cannot have a dc component, so the first term cannot contribute either. Therefore, only the "pairs" terms contribute to the fourth moment.

Two features of the narrow-band case immediately become apparent. First, the band can be translated rigidly on the frequency axis because if a constant is added to all the integers n, j, k, and l, the equalities among sums of pairs are unchanged. Second, it is clear that a constant can be added to all the phase angles because a common phase cancels in the argument of the cosine in the "pairs" terms. These are the features of the translatable "trenchlike" solutions described in Sec. III. These features actually occur for any spectrum where the "3-1" terms do not exist. Except for narrow bands, however, such spectra are rather peculiar cases. For instance, if one begins with harmonics 1 and 2, then the lowest-order three-component signal has harmonics 1,2,7. The next is 1,2,7,8; then 1,2,7,8,13, and so on. Allowed spectra have large gaps.

3. Equal amplitude, common phase

If all amplitudes are equal ($A_n=1$ for all $n\leqslant N$) and all phases are the same ($\phi_n=\phi$ for all n) then some simplifications occur in Eq. (A4): All cosine factors in the "pairs" terms become equal to 1, and all cosine factors in the "3-1" terms become equal to $\cos(2\phi)$. Evaluating the fourth moment then becomes a matter of counting the number of terms of each type, for a given number N of nonzero harmonics. For instance, for the "pairs" type, we must count the number of ways that one can choose two numbers from a set and have their sum equal the sum of two other numbers from the set, where the set contains the integers from 1 to N inclusive. Any integer can be used any number of times.

Normalizing by the square of the power, which equals $N^2/4$, we find

$$W=\frac{1}{N}\left(N^2+\frac{1}{2}\right)+\frac{(N-1)(N-2)}{3N}\cos(2\phi). \tag{A8}$$

Whatever the angle ϕ, the moment W continues to grow, linearly with increasing N for large N, as shown in Fig. 1. The worst choice is $\phi=0$, corresponding to a waveform that is a sum of cosines. The best choice is $\phi=-\pi/2$, corresponding to a sum of sines, but that does not make it a good choice. For large N, any fixed-phase waveform has a larger fluctuation than the average random-phase waveform.

4. Octave spectra

This section proves that when all the harmonics of a tone are octaves of a fundamental, then the fourth moment of the tone is minimized by phases that are independent of the amplitudes of the harmonics. A result of this fact is that if an optimized waveform is passed through a dispersionless filter, or other process that alters only the spectral envelope, then it remains an optimized waveform.

The independence comes from the fact that the octave spectrum has such large gaps that only a few combinations of indices satisfy the requirements imposed by the Kronecker deltas. There are no "pairs" terms with any angular dependence, and the "3-1" terms are of a simple form. The fourth moment is given by

$$\overline{x^4}=\frac{3}{8}\sum_n A_n^4+\frac{3}{4}\sum_{n,j;n\neq j}A_n^2A_j^2+\frac{3}{2}\sum_n A_n^2A_{2n}A_{4n}\times\cos(2\phi_n+\phi_{2n}-\phi_{4n}). \tag{A9}$$

Equation (A9) shows that there is only a single kind of term that includes phases. Like any fourth power, the above function must be positive, and therefore, the function is made as small as possible by making such terms as negative as possible. This occurs when the sum of the phases in parentheses is π. One solution is to let all phase angles be equal to $-\pi/2$, which corresponds to summing sine waves. There are other solutions too: for example, alternating phases $\{0,\pi,0,\pi,...\}$ is a solution corresponding to a series of cosines with alternating sign. The general solution has the form

$$\{\phi_1,\phi_2,\phi_4,\phi_8,...\}=(\pi/2)\{1,1,1,1,...\}+D\{1,-1,1,-1,...\}, \tag{A10}$$

where D is an arbitrary number.

The actual value of the fourth moment depends upon the amplitudes. If all amplitudes are equal up to harmonic number N and zero thereafter, then the normalized fourth moment becomes

$$W=\tfrac{3}{2}[(2N^2-5N+8)/N^2],\quad \text{for } N>1. \tag{A11}$$

Equation (A9) shows that the phase dependence arises from a minimum of three successive octave components with nonzero amplitudes. For an octave spectrum with gaps, e.g., harmonics 1,2,8,16,64, the fourth moment has no phase dependence at all.

5. Equal amplitude, random phases

Equation (A4) can be used in a simple derivation of the ensemble-averaged fourth moment for the case of equal amplitudes and random phases. The ensemble average of each cosine term in (A4) is zero unless there is some special relationship among the indices of the sum. For example, in the quadruple sum over n, j, k, and l there is the special case where all indices are equal. Then the cosine of the phase differences in the "pairs" terms is one and does not average to zero.

There is no special relationship that permits the "3-1" terms to contribute; therefore they always average to zero. The finite contributions all come from the "pairs" terms, and the most important of these occur when indices are equal in pairs. Therefore, the unnormalized fourth moment becomes

$$\langle \overline{x^4} \rangle = \frac{3}{8} \left(\sum_{n,j=1}^{N} 1 + \sum_{n,j=1}^{N} 1 - \sum_{n=1}^{N} 1 \right). \tag{A12}$$

The first term in the large parenthesis comes when $k = n$ and $l = j$. The second term comes when $k = j$ and $l = n$. The third term corrects for the fact that by adding up the first and second terms we have double counted the case where all indices are equal.

The sums are easy to do. The first two are N^2, and the third is N. Then, normalizing by the square of the signal power, we find that the ensemble-averaged fourth moment is

$$\langle W \rangle = 3[1 - 1/(2N)]. \tag{A13}$$

This expression agrees with the fourth moment derived in Appendix A of Hartmann and Pumplin (1988), but the derivation here is simpler, given Eq. (A4). It agrees well with the square symbols shown in Fig. 1, found by averaging the fourth moments from 10 000 waveforms. The large N limit, $W = 3$, is just what is expected from the central limit theorem. When the number of components becomes large, the distribution of instantaneous values becomes a Gaussian. The fourth moment of a normalized Gaussian is 3.

6. Equal amplitude, alternating phases

The waveform composed of alternating sine and cosine functions, used by Plomp and others, is a special case of the more general "alternating-phases" waveform. We have derived analytical results for the case of alternating phases for spectra that are similar to those in Sec. A 3 above, namely all harmonics have the same amplitude up to harmonic number N, and beyond that number all amplitudes are zero. The technique is straightforward: We write Eq. (A4) for the special case of two alternating phase angles, and set the derivatives with respect to those two variables equal to zero. The resulting equations have several solutions and we then test to see which leads to the smallest fourth moment.

In the end, we find that the alternating phases that minimize the fourth moment depend upon whether N is even or odd and also upon the range of N. For N even and $N \leqslant 12$, the lowest fourth moment, for alternating phases, occurs when

$$\{\phi_1, \phi_2, \phi_3, \phi_4, \ldots\} = \{\pi/2, 0, \pi/2, 0, \ldots\}, \tag{A14}$$

corresponding to an alternating sine/cosine series. The fourth moment is

$$W = (N^2 - 2N + 9)/(2N). \tag{A15}$$

For N odd and $N \leqslant 7$, the smallest fourth moment, for alternating phases, again occurs for the alternating sine/cosine series, and the fourth moment is

$$W = (N^3 - N^2 + 4N - 1)/(2N^2). \tag{A16}$$

For larger values of N (larger than 12 or 7) the optimum pair of alternating phases does not correspond to an alternating sine/cosine series. The analytic expression for the fourth moment also becomes complicated. Of interest, though, is the fact that in the limit of large N, the fourth moment grows linearly with N with a slope of 4/9. This can be compared with the large-N behavior for the best common phase, namely sine phase ($\phi = \pi/2$), where the slope is 2/3.

Thus, for large N, the fluctuation for the best alternating-phase signal is smaller than the fluctuation for the best single-phase signal. But for large N, even the best alternating-phase signal is bound to be worse than the average random-phase signal, where the fourth moment goes to a finite limit of 3.

[1] For a waveform with a fundamental and a third harmonic, the minimum peak factor and the minimum fourth moment actually both occur when the third harmonic phase angle is 180 deg with respect to the fundamental phase, regardless of the relative amplitudes of the harmonics. Naively applying Schroeder's algorithm, one finds that the relative phase depends upon harmonic amplitudes, which is incorrect. If the two amplitudes are equal, that algorithm leads to a relative phase of zero, the worst possible choice.

[2] It is not hard to show that minimizing the fourth moment minimizes the power fluctuation. Let P be the instantaneous waveform power. The variance in power is the average of P^2 minus the square of the average P, where averages are computed over a period of the waveform. The average P does not depend upon phase angles; it depends only upon harmonic amplitudes. Therefore, a choice of phases that minimizes the average of P^2 is the choice that minimizes the variance in power. Instantaneous power P itself is equal to the square of the waveform, i.e., to $x^2(t)$, given the usual assumption of a unit resistive impedance. Therefore, P^2 is given by $x^4(t)$, and this is the quantity that must be minimized.

[3] See AIP document number PAPS-JASMA-90-1986-15 for 15 pages of Minimum Power Fluctuation Extended Tables (MPFET). Order by PAPS number and journal reference from the American Institute of Physics, Physics Auxiliary Publication Service, 335 East 45th Street, New York, NY 10017. The price is \$1.50 for a microfiche or \$5.00 for a photocopy. Airmail is additional. Make checks payable to the American Institute of Physics.

[4] Phase freedoms were used in the construction of the tables in this article. Arbitrarily, we required that the phase of the lowest frequency component should be zero and the phase of the next lowest component should be between 1.5π and 2π. Removing phase arbitrariness with a two-part convention of this kind, enabled us to recognize equivalent minima when they appeared in the output of the algorithm.

[5] Our calculations for "dark" signals were done for $q = 2$ and 3 and for all possible numbers of harmonics from 3 to 50; i.e., $N_1 = 1$ and N_2 ranging from 3 to 50. Each calculation employed 100 starts. For $q = 2$, our calculation never found a local minimum with a fourth moment less than that for the alternating sin, cos, sin... series. For $q = 3$, our calculation never found any local minimum at all other than the alternating sin, cos, sin...series. It seems evident that for q equal to about 3 or larger the fourth moment is dominated by the low- order harmonics so that adding higher harmonics to the spectrum only creates inflection points and not local minima in the fourth moment function.

[6] For the pseudo-half-wave, and for the pseudotriangle of the preceding section, we performed calculations with a finite number of harmonics with $N_1 = 1$ and N_2 taking on all values between 4 and 50 (5 and 51 for the triangle). Out of 100 starts, the waveform given by the pseudowaveform of

the text had the smallest fourth moment solution for all values of N_2. Only a few starts led to local minima other than the minimum of the pseudo-waveform.

[7] This argument shows that a square wave has only odd harmonics. A pulse with $p = 1/7$ is reminiscent of the mythical piano in which the hammer strikes the string at a point that is exactly 1/7th of the string's length. (Hall and Clark, 1987).

[8] In order to be confident that our lowest minimum is the global minimum, it is necessary to fall into all of the holes in the space. But with starting points that are independent and random, our confidence can only be based upon statistics. We have two kinds of evidence. First is simply the weight of numbers. If we find that many different starting points in the space all lead to the same few holes then we suppose that we have found all the holes. The second kind of evidence is based upon the size of holes. It is possible to estimate the sizes of holes and the way that they fill the volume of the space: Starting from the bottom of a hole, we take steps of various lengths in random directions to find a set of test points. We follow the gradient at each test point. If it leads back into the same hole then the test point is counted as part of the hole volume. By this technique we learn that different holes have rather similar volumes. By dividing the total volume of the space by the volume of a typical hole we find an estimate of the number of holes in the space, and this too helps to decide whether we have found all the holes or not.

Allen, J. B. (1983). "Magnitude and phase-frequency response to single tones in the auditory nerve," J. Acoust. Soc. Am. **73**, 2071–2092.

Buunen, T. J. F. (1976). "On the perception of phase differences in acoustic signal," doctoral thesis, University of Delft, Delft, The Netherlands.

de Boer E. (1961). "A note on phase distortion and hearing," Acustica **11**, 182–184.

Deng, L., Geisler, C. D., and Greenberg, S. (1987). "Responses of auditory- nerve fibers to multiple-tone complexes," J. Acoust. Soc. Am. **82**, 1989–2000.

Hall, D. E., and Clark, P. (1987). "Piano string excitation IV: The question of missing modes," J. Acoust. Soc. Am. **82**, 1913–1918.

Hartmann, W. M. (1987). "Temporal fluctuations and the discrimination of spectrally dense signals by human listeners," in *Auditory Processing of Complex Sounds*, edited by W. A. Yost and C. S. Watson (Erlbaum, Hillsdale, NJ).

Hartmann, W. M., and Pumplin, J. (1988). "Noise power fluctuations and the masking of sine signals," J. Acoust. Soc. Am. **83**, 2277–2289.

Klatt, D. H. (1980). "Software for a cascade/parallel formant synthesizer," J. Acoust. Soc. Am. **67**, 971–995.

Lehr, A. (1987). "From theory to practice," Music Percep. J. **4**, 267–280. (This issue includes three other articles on the major bell.)

Patterson, R. D. (1987). "A pulse ribbon model of monaural phase perception," J. Acoust. Soc. Am. **82**, 1560–1586.

Plomp, R. (1989). "The perception of timbre of steady-state complex tones," J. Acoust. Soc. Am. Suppl. 1 **86**, S67.

Plomp, R., and Steeneken, H. J. M. (1969). "Effect of phase on the timbre of complex tones," J. Acoust. Soc. Am. **46**, 409–421.

Preece, J. P., and Wilson, R. H. (1988). "Detection, loudness and discrimination of five-component tonal complexes differing in crest factor," J. Acoust. Soc. Am. **84**, 166–171.

Pumplin, J. (1985). "Low-noise noise," J. Acoust. Soc. Am. **78**, 100–104.

Schroeder, M. R. (1970). "Synthesis of low-peak-factor signals and binary sequences with low autocorrelation," IEEE Trans. Inf. Theory **16**, 85–89.

Shepard, R. N. (1964). "Circularity in judgements of relative pitch," J. Acoust. Soc. Am. **36**, 2346–2353.

Singh, P. (1987). "Perceptual organization of complex tone sequences: A tradeoff between pitch and timbre," J. Acoust. Soc. Am. **82**, 886–899.

Smith, B. K., Sieben, U. K., Kohlrausch, A., and Schroeder, M. R. (1986). "Phase effects in masking related to dispersion in the inner ear," J. Acoust. Soc. Am. **80**, 1631–1637.

Tomlinson, R. W. W., and Schwartz, D. W. F. (1988). "Perception of the missing fundamental in nonhuman primates," J. Acoust. Soc. Am. **84**, 560–565.

Ueda, K., and Ohgushi, K. (1987). "Perceptual components of pitch: Spectral representation using a multidimensional scaling technique," J. Acoust. Soc. Am. **82**, 1193–1200.

Zatorre, R. J. (1988). "Pitch perception of complex tones and human temporal lobe function," J. Acoust. Soc. Am. **84**, 566–572.