QCD thermodynamics on the lattice:
(approaching the continuum limit
with physical quark masses)

Z. Fodor

University of Wuppertal

results of the Wuppertal-Budapest group

Nuclear Dynamics 2010, Ocho Rios, Jamaica
Bielefeld-Brookhaven-Riken-Columbia Coll. (+MILC=‘hotQCD’):

$T_c$ from both $\chi_{\bar{\psi}\psi}$ and Polyakov loop:

$T_c = 192(7)(4)$ MeV

Wuppertal-Budapest group (WB):

chiral susceptibility:

$T_c = 151(3)(3)$ MeV

Polyakov and strange susceptibility:

$T_c = 175(2)(4)$ MeV

‘chiral $T_c$’: $\approx 40$ MeV; ‘confinement $T_c$’: $\approx 15$ MeV difference

both groups give continuum extrapolated results with physical $m_\pi$
Literature: discrepancies between T dependences

Reason: shoulders, inflection points are difficult to define?  
Answer: no, the whole temperature dependence is shifted

for chiral quantities \( \approx 35 \text{ MeV} \); for confinement \( \approx 15 \text{ MeV} \)
this discrepancy would appear in all quantities (eos, fluctuations)

150 MeV transition temperature: isn’t it a bit too small?  
lattice works in \( V \to \infty \), which gives much smaller \( T_c \).
Safe and unsafe lattice approaches

three very important issues for a safe lattice result:
i. discretization effects; ii. continuum limit; iii. physical quark masses

i. discretization effects: example of the $m_{ud} - m_s$ phase diagram

$n_f=3$ case (standard action, $N_t=4$): critical $M_{ps} \approx 300$ MeV
different discretization effects (stout or p4 action or $N_t=6$):
critical $M_{ps}$ drops even below 100 MeV (physical $M_{ps}$ is between)
$\implies$ huge uncertainties: safe only in the continuum limit

note: state of the art $T=0$ physics $a<0.06$ fm; $T>0$ physics $a>0.1$ fm

F. Karsch et al., Nucl. Phys. Proc. 129 ('04) 614; G. Endrodi et al. PoS Lat'07 182 ('07); de Forcrand et al., ibid 178
Safe and unsafe lattice approaches

ii. continuum limit: is there any way to show that it is safe?
   a. monitor asymptotic scaling \((a^2)\)
   b. with different scale settings

\(N_t=4,6\): inconsistent, but \(N_t=6,8,10\) consistent continuum limit independently of the quantity used to set the physical scale (MeV)

iii. physical quark masses are needed
   (non-physical masses: only dimensionless ratios in the continuum)
   pion splitting (only one \(\pi\) instead of three) \(\propto a^2 \Rightarrow\) mimics large \(M_\pi\)
   some actions are quite good at reducing this non-physical effect
Safe and unsafe lattice approaches

ii. continuum limit: don’t depend on the quantity which fixes the scale
monitor asymptotic scaling ($a^2$) with different scale settings

$N_t=4,6$: inconsistent, but $N_t=6,8,10$ consistent continuum limit
independently of the quantity used to set the physical scale (MeV)

iii. physical quark masses are needed to give results in MeV
(non-physical masses: only dimensionless ratios in the continuum)
pion splitting (only one $\pi$ instead of three) $\propto a^2 \Rightarrow$ mimics large $M_\pi$
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QCD thermodynamics on the lattice: (approaching the continuum)
Finite-size scaling theory

problem with phase transitions in Monte-Carlo studies
Monte-Carlo applications for pure gauge theories ($V = 24^3 \cdot 4$)
extistence of a transition between confining and deconfining phases:
Polyakov loop exhibits rapid variation in a narrow range of $\beta$

- theoretical prediction: SU(2) second order, SU(3) first order
  $\Rightarrow$ Polyakov loop behavior: SU(2) singular power, SU(3) jump

data do not show such characteristics!

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Finite size scaling in the quenched theory

look at the susceptibility of the Polyakov-line first order transition (Binder) $\Rightarrow$ peak width $\propto 1/V$, peak height $\propto V$

finite size scaling shows: the transition is of first order
The nature of the QCD transition


finite size scaling study of the chiral condensate (susceptibility)

\[ \chi = \left( \frac{T}{V} \right) \frac{\partial^2 \log Z}{\partial m^2} \]

phase transition: finite V analyticity \( V \to \infty \) increasingly singular
(e.g. first order phase transition: height \( \propto V \), width \( \propto \frac{1}{V} \))
for an analytic cross-over \( \chi \) does not grow with \( V \)

two steps (three volumes, four lattice spacings):
a. fix \( V \) and determine \( \chi \) in the continuum limit: \( a=0.3, 0.2, 0.15, 0.1 \) fm
b. using the continuum extrapolated \( \chi_{max} \): finite size scaling
Approaching the continuum limit

\[ a = 0.3 \text{ fm} \]

3.6 fm  4.8 fm  6 fm
Approaching the continuum limit

\[ a = 0.2 \text{ fm} \]

\[ 3.6 \text{ fm} \quad 4.8 \text{ fm} \quad 6 \text{ fm} \]
Approaching the continuum limit

\[ a = 0.15 \text{ fm} \]

3.6 fm  \quad 4.8 fm  \quad 6 \text{ fm}

\[ 1/N_t^2 \propto a^2 \quad 1/N_t^2 \propto a^2 \quad 1/N_t^2 \propto a^2 \]
Approaching the continuum limit

\[ a = 0.12 \text{ fm} \]

3.6 fm  4.8 fm  6 fm

\[ \frac{1}{N_s^2} \propto a^2 \]

\[ \frac{1}{N_t^2} \propto a^2 \]

\[ \frac{1}{N_t^2} \propto a^2 \]
Approaching the continuum limit

3.6 fm  

4.8 fm  

6 fm

\[ T^4 / (m^2 a^2) = \begin{cases} \frac{1}{N_s^2} \propto a^2 & N_s/N_t = 3 \\ \frac{1}{N_t^2} \propto a^2 & N_s/N_t = 4 \\ \frac{1}{N_t^2} \propto a^2 & N_s/N_t = 5 \end{cases} \]
The nature of the QCD transition: result

- finite size scaling analysis with continuum extrapolated $T^4/m^2 \Delta \chi$

\[
\frac{T^4}{m^2 \Delta \chi}
\]

the result is consistent with an approximately constant behavior for a factor of 5 difference within the volume range

chance probability for $1/V$ is $10^{-19}$ for $O(4)$ is $7 \cdot 10^{-13}$

continuum result with physical quark masses in staggered QCD:

the QCD transition is a cross-over
Analytic QCD transition (cross-over): consequences

highly non-trivial $a \rightarrow 0$ result

analytic transition (cross-over) $\Rightarrow$ it has no unique $T_c$:
examples: melting of butter (not ice) & water-steam transition

above the critical point $c_p$ and $\frac{d\rho}{dT}$ give different $T_c$s.

QCD: chiral & quark number susceptibilities or Polyakov loop
they result in different $T_c$ values $\Rightarrow$ physical difference
Discretization errors in the transition region

we always have discretization errors: nothing wrong with it as long as
a. result: close enough to the continuum value (error subdominant)
b. we are in the scaling regime ($a^2$ in staggered)

various types of discretization errors $\Rightarrow$ we improve on them (costs)

we are speaking about the **transition temperature region**

**interplay** between hadronic and quark-gluon plasma physics

smooth cross-over: one of them takes over the other around $T_c$

both regimes (low T and high T) are equally important

improving for one: $T \gg T_c$, doesn’t mean improving for the other: $T < T_c$

example: ’expansion’ around a Stefan-Boltzman gas (van der Waals)

for water: it is a fairly good description for $T \gtrsim 300^\circ$

calculate the boiling point: more accuracy needed for the liquid phase
Examples for improvements, consequences

analytic: how to reach the continuum free energy density at $T=\infty$?

Bielefeld: p4 action is essentially designed for this quantity $T \gg T_c$

MILC: asqtad is designed mostly for $T=0$ (but good at $T \gg T_c$, too)

BW: stout-smeared link converges slower but in the $a^2$ scaling regime (e.g. extrapolation from $N_t=8,10$ provides a result within about 1%)
Chiral symmetry breaking and pions

transition temperature for remnant of the chiral transition: balance between the chirally broken and chirally symmetric sectors
chiral symmetry breaking: 3 pions are the pseudo-Goldstone bosons
staggered QCD: 1 pseudo-Goldstone instead of 3 (taste violation)
staggered lattice artefact $\Rightarrow$ disappears in the continuum limit
WB: stout-smeared improvement is designed to reduce this artefact
in lattice QCD we use $g, m_{ud}$ and $m_s$ in the Lagrangian ('a' not) measure e.g. the vacuum mass of a hadron in lattice units: $M_\Omega a$
since we know that $M_\Omega = 1672$ MeV we obtain 'a' and $T = 1/N_t a$


independently which quantity is taken (we used physical masses)

⇒ one obtains the same 'a' and $T$, result is safe
T>0 results: strange susceptibility

Compare the Wuppertal-Budapest results [Y.Aoki et al. JHEP 0906:088,2009] with ‘hotQCD’

‘hotQCD’ results are on $N_t=8$, WB results are on $N_t=8,10,12,(16)$

‘hotQCD’: results with two different actions are almost the same

WB: for large T one extrapolates according to the known $a^2$ behaviour

WB: no change in the lattice results compared to our 2006 paper

note, that the experimental value of $f_K$ decreased by 3% since 2006

about 20 MeV difference between the results
T>0 results: chiral condensate

Compare the Wuppertal-Budapest results [Y. Aoki et al. JHEP 0906:088, 2009] with ‘hotQCD’

‘hotQCD’ results are on $N_t=8$, WB results are on $N_t=8,10,12$

‘hotQCD’: results with two different actions are almost the same

WB: no lattice spacing dependence observed for $N_t=8,10,12$

WB: no change in the lattice results compared to our 2006 paper

about 35 MeV difference between the results
transition temperatures for various observables

<table>
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<th>$\chi_{\bar{\psi}\psi}/T^4$</th>
<th>$\chi_{\bar{\psi}\psi}/T^2$</th>
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<th>$\Delta_{I,s}$</th>
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renormalized chiral susceptibility, renormalized chiral condensate
Polyakov loop and strange quark number susceptibility

no change compare to our 2006 data (errors are reduced)

note, that the experimental value of $f_K$ decreased by 3% since 2006

Particle Data Group now gives $f_K = 155.5(2)(8)(2)$ MeV (error 0.5%)

$r_0$ is not directly measurable:

ETM: $0.444(4)$ fm, QCDSF: $0.467(6)$ fm,
HPQCD&UKQCD: $0.469(7)$ fm, PACS-CS: $0.492(6)(+7)$ fm
Illustration: lattice artefacts due to pion splitting

we have seen: our action (WB) has less unphysical pion splitting than the asqtad (MILC) and far less than the p4 (Bielefeld) action in the continuum limit: no problem; at $a \neq 0$ it mimics larger $M_\pi$ “reproduce” the result of hotQCD with larger $M_\pi$ (asqtad is better)

$M_\pi \approx 220$ MeV (hotQCD) “corresponds” to $M_\pi \approx 410$ MeV (WB) asqtad (MILC) needs finer p4 (Bielefeld) needs much finer lattices in order to handle physical quark masses (suboptimal improvement)
go even closer to the continuum limit: $N_t=16$ (preliminary)
compare WB and hotQCD with the hadron resonance gas model

Wuppertal-Budapest: **nice agreement** upto 160 MeV (almost $T_c$)
hot-QCD: **several sigma discrepancy**
reason: hotQCD action has large $\pi$-splitting $\Rightarrow$ large hadron masses
Story of $T_c$ (last 4 years) hotQCD versus Wuppertal

Results in physical units (MeV) are meaningful only in the continuum limit (extrapolation) with physical quark masses.

2006 claims: continuum limit results with physical quark masses many standard deviation discrepancy: statistically “impossible” our interpretation since 2006: BBRC result on $T_c$ is “non-continuum” hotQCD preliminary: closer to continuum $\Rightarrow$ more than 20 MeV drop trust a result: scaling behavior & independence of scale setting
our improved action: Symanzik gauge & stout smeared fermions 
physical quark masses and our line of constant physics can be used renormalization: G.Endrodi, Z.Fodor, S.D.Katz, K.K.Szabo, PoS Lat07 228 (2007) 
leading order (SB limit) normalization is also applied full temperature dependence up to 1 GeV is given for $N_t=8$ (monitor the continuum limit: soon 3 sensitive points with $N_t=10,12$)

hotQCD result has a larger characteristic temperature (similar to $T_c$) 
hotQCD indicates a steeper (more singular) transition (similar to $T_c$)
Equation of state: preliminary results

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Introduction
The nature of the QCD transition
QCD transition temperature
The QCD equation of state
Summary

Summary

- QCD transition: analytic cross-over (continuum with $M_\pi = 135$ MeV)
- new (2009) results for the transition temperature
- three improvements since 2006
  a. at T=0 all simulations are done with physical quark masses
  b. to verify that the results are independent of the scale setting we use 5 experimentally well-known quantities: $f_K, f_\pi, m_{K^*}, m_\Omega, m_\Phi$
  c. even smaller lattice spacings: $N_t=12$ (in one case $N_t=16$)
- all findings are in complete agreement with our 2006 results
- Particle Data Group reduced the experimental value of $f_K$: 3%
- discrepancy between Wuppertal-Budapest & 'hotQCD' results
  a. for the remnant of the deconfinement transition: about 20 MeV
  b. for the remnant of the chiral transition: about 35 MeV
- possible explanation: hotQCD has large discretization effects
- preliminary equation of state results $N_t=8(10,12)$
**Lattice Lagrangian: gauge fields**

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + \bar{\psi} (D_\mu \gamma^\mu + m) \psi \]

anti-commuting \( \psi(x) \) quark fields live on the sites

gluon fields, \( A^a_\mu(x) \) are used as links and plaquettes

\[
U(x, y) = \exp \left( i g_s \int_x^y dx'^\mu A_\mu^a(x') \lambda_a / 2 \right)
\]

\[
P_{\mu\nu}(n) = U_\mu(n) U_\nu(n + e_\mu) U_\mu^\dagger(n + e_\nu) U_\nu^\dagger(n)
\]

\( S = S_g + S_f \) consists of the pure gluonic and the fermionic parts

\[
S_g = 6 / g_s^2 \cdot \sum_{n,\mu,\nu} \left[ 1 - \text{Re}(P_{\mu\nu}(n)) \right]
\]

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QCD thermodynamics on the lattice: (approaching the continuum)
Lattice Lagrangian: fermionic fields

quark differencing scheme:

$$\bar{\psi}(x) \gamma^\mu \partial_\mu \psi(x) \rightarrow \bar{\psi}_n \gamma^\mu (\psi_{n+e_\mu} - \psi_{n-e_\mu})$$

$$\bar{\psi}(x) \gamma^\mu D_\mu \psi(x) \rightarrow \bar{\psi}_n \gamma^\mu U_\mu(n) \psi_{n+e_\mu} + \ldots$$

fermionic part as a bilinear expression: $$S_f = \bar{\psi}_n M_{nm} \psi_m$$

we need 2 light quarks (u,d) and the strange quark: $$n_f = 2 + 1$$

(complication: fermion doubling ⇒ staggered/Wilson)

Euclidean partition function gives Boltzmann weights

$$Z = \int \prod_{n,\mu} [dU_\mu(x)][d\bar{\psi}_n][d\psi_n] e^{-S_g - S_f} = \int \prod_{n,\mu} [dU_\mu(n)] e^{-S_g} \det(M[U])$$
Importance sampling

\[ Z = \int \prod_{n,\mu} [dU_\mu(n)] e^{-S_g} \det(M[U]) \]

we do not take into account all possible gauge configuration

each of them is generated with a probability \( \propto \) its weight

importance sampling, Metropolis algorithm:
(all other algorithms are based on importance sampling)

\[ P(U \rightarrow U') = \min [1, \exp(-\Delta S_g) \det(M[U'])/\det(M[U])] \]

gauge part: trace of 3\(\times\)3 matrices (easy, without M: quenched)

fermionic part: determinant of 10\(^6\) \(\times\) 10\(^6\) sparse matrices (hard)

more efficient ways than direct evaluation (Mx=a), but still hard
$T_c$ strongly depends on the geometry

nanotube-water doesn’t freeze, even at hundred degrees below $0^\circ$C


use ‘confined’ spatial boundary conditions: more like experiments

large deviation (upto 30 MeV) from the infinite volume limit

if $V \to \infty$ is 150 MeV a 100 fm$^3$ system might have 170 MeV
scaling regime is reached if $a^2$ scaling is observed asymptotic scaling starts only for $N_t > 8$ ($a \lesssim 0.15 \text{ fm}$): two messages

a. $N_t=8,10$ extrapolation gives 'p' on the $\approx 1\%$ level: good balance
b. stout-smeared improvement is designed to reduce this artefact

most other actions need even smaller 'a' to reach scaling
Scaling of $B_K$ in quenched simulations


$B_K^{\text{NDR}}$ (2 GeV) in the quenched approximation

- Unimproved (JLQCD)
- Unimproved (this work)
- HYP
- Asqtad (Wilson glue)

Unimproved action has large scaling violations
Asqtad action is somewhat better
HYP smeared improvement $\Rightarrow$ almost perfect scaling