Postulates of Group Theory

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1. A **group** is a set of abstract elements $g \in \{a, b, c, \ldots$ for which there is a single composition law, $\circ$, (normally called "multiplication") which satisfies the following four postulates:

   (a) If a and $b \in \mathcal{G}$ and $c = a \circ b$, then $c \in \mathcal{G}$. This is closure, or sometimes called the "group property".

   (b) $(a \circ b) \circ c = a \circ (b \circ c)$

   (c) $\exists g_i \equiv e$ such that $e \circ g_i = g_i \circ e = g_i$. This insures the existence of the identity.

   (d) $\forall_{g_i \in \mathcal{G}} \exists g'$ such that $g \circ g' = g' \circ g = e$. This insures the existence of the inverse for all elements.

2. The **order** of a group is the number of elements ... it can be finite or infinite.

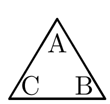
3. **Abelian Groups** are those in which the elements commute.

4. **Discrete Groups** are those which have a countable, finite order.

5. Two groups which have the same multiplication table are **isomorphic** ... a $1:1$ mapping exists between the elements.

6. If, from a group $\mathcal{G}$, a subset of elements, $\mathcal{H}$, can be selected which itself forms a group having the same combinatoric law of $\mathcal{G}$, then $\mathcal{H}$ is a **subgroup** of $\mathcal{G}$.

7. The Dihedral Group, D_3 in three dimensions can be realized as the set of rotations of an equilateral triangle.



The elements $\{a, b, c\}$ are rotations by π through an axis through the vertices A, B, C and the center of the triangle. The elements $\{d, f\}$ are rotations in the plane of the triangle of $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$, clockwise about the center of the triangle. The multiplication table for D_3 is:

$\circ$	e	a	b	c	d	f
e	e	a	b	c	d	f
a	a	e	f	d	c	b
b	b	d	e	f	a	c
c	c	f	d	e	b	a
d	d	b	c	a	f	e
f	f	c	a	b	e	d

8. Two groups $\mathcal{G}$ and $\mathcal{H}$ are **homomorphic** if some $h_1 \in \mathcal{H}$ can be associated wtih each element in $\mathcal{G}$ such that if $g_1 \circ g_2 = g_3 \in \mathcal{G}$ that $h_1 \circ h_2 = h_3 \in \mathcal{H}$.

9. A **complex** is a set of elements from a group.

10. Within a group $\mathcal{G},\ g_i \in \mathcal{G}$ and $g_j \in \mathcal{G}$ are conjugate elements if there exists some $g_h \in \mathcal{G}, \ni g_i = g_h \circ g_j \circ g_h^{-1}$.

11. Elements which are conjugates to one another are together elments of a **class**. Each element belongs to only one class.

12. If $\mathcal{H}$ is a subgroup of $\mathcal{G}$ and $g \in \mathcal{G}$ then $\mathcal{H}' = \{g \circ h \circ g^{-1};\ h \in \mathcal{H}\}$ also forms a subgroup of $\mathcal{G}$, $\mathcal{H}'$ is a **conjugate subgroup** of $\mathcal{G}$.

13. An **invariant subgroup** is a subgroup, $\mathcal{H}$ of $\mathcal{G}$ which is identical to all of its conjugate subgroups.

14. A group is **simple** if it does not contain any nontrivial invariant subgroups. A group is **semisimple** if it does not contain any Abelian invariant subgroups.

15. Let $\mathcal{H} = \{h_1, h_2, h_3, \dots\}$ be a subgroup of $\mathcal{G} = \{g\}$ where $\{g'\}$ is a complex of $\mathcal{G}$ not in $\mathcal{H}$. Then $\{g' \circ h_1, g' \circ h_2, \dots\}$ is called a **left-coset** of $\mathcal{H}$. $\{h_1 \circ g', h_2 \circ g', \dots\}$ is called a **right-coset** of $\mathcal{H}$. These are not subgroups.

16. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be subgroups of $\mathcal{G}$ with:

    (a) one element in common,

    (b) every element of $\mathcal{H}_1$ commuting with every element of $\mathcal{H}_2$, and

    (c) every element of $\mathcal{G}$ can be written as $g = h_1 \circ h_2$,

    then $\mathcal{G}$ is a direct product group, $\mathcal{G} = \mathcal{H}_1 \otimes \mathcal{H}_2$.

17. A **representation** of a group is a mapping of the elements of $\mathcal{G}$ onto a group of linear operators defined in a linear vector space, $\mathcal{V}$.

18. When it is possible to put a matrix representation of a group into block diagonal form, it is called a **reducible representation**. If the submatrices are not capable of being put themselves into further block diagonal form, they are the **irreducible represenations**, IRR. There are as many IRR as there are classes for a group.

19. If two represenations of the same dimension are related by

$$\Gamma_{(i)}(g_k) = \mathbb{B}^{-1}\Gamma_{(j)}(g_k)\mathbb{B},$$

then $\Gamma_{(i)}$ and $\Gamma_{(j)}$ are equivalent. If no $\mathbb{B}$ exists, then the representations are inequivalent.