Postulates of Quantum Mechanics

PHY 853, Fall Semester, 1999

Chip Brock, brockchip.pa.msu.edu

October 4, 1999

I. The state of a quantum mechanical system is exhaustively represented at any time by a vector in a normed Hilbert Space. This vector is called the *state vector***.**

Definition 1 (scalar) *Any abstract entity satisfying the following primary combinatoric relations:*

Postulate 1 (Addition) *To every pair, a and b of scalars which are in a set, S, there corresponds another scalar $a \oplus b$ called the sum or addition of a and b.*

P 1.1 (closure) *for every pair of elements a and b, the sum $a \oplus b$ is in S.*

P 1.2 (commutivity) *$a \oplus b = b \oplus a$ is the commutative law of scalar addition;*

P 1.3 (associativity) *$a \oplus (b \oplus c) = (a \oplus b) \oplus c$ is the associative law of scalar addition;*

P 1.4 (zero) *there exists the element 0 such that $a \oplus 0 = a$ defines a unique zero; and*

P 1.5 (inverse) *there exists a for each a, an a' such that for each a, $a \oplus a' = \varnothing$ which defines the inverse or negative for each scalar.*

Postulate 2 (Product) *To every pair, a and b of scalars, there corresponds another scalar $a \otimes b$ called the product or multiplication of a and b.*

P 2.1 (closure) *for every pair of elements a and b, the sum $a \otimes b$ is in S.*

P 2.2 (commutativity) *$a \otimes b = b \otimes a$ is the commutative law of scalar multiplication;*

P 2.3 (associativity) *$a \otimes (b \otimes c) = (a \otimes b) \otimes c$ is the associative law of scalar multiplication;*

P 2.4 (identity) *There exists a $\mathcal{I}$ such that $a \otimes \mathcal{I} = a$ defines a unique identity;*

P 2.5 (inverse) *There exists an* a'' *such that for all* a, $a \otimes a'' = \mathcal{I}$ *which defines the inverse* ($a'' \equiv a^{-1}$) *for each scalar; and*

P 2.6 (distributivity) $a \otimes (b \oplus c) = a \otimes b \oplus a \otimes c$ *is the distributive law between the two operations.*

Definition 2 *Any set of symbols which satisfies the above conditions is formally a Field. Examples of fields include the set of all real numbers,* $\mathcal{R}$ *and the set of all complex numbers,* $\mathcal{C}$.

Definition 3 *Over a scalar field,* $\mathcal{F}$, *we can define a Vector Space,* $\mathcal{V}$, *as a collection of symbols which satisfy a set of conditions.*

Postulate 3 (Vector Sum) *To every pair of vectors,* $|\xi\rangle$ *and* $|\eta\rangle$ *in* $\mathcal{V}$, *there is another vector* $|\xi\rangle \oplus |\eta\rangle$ *called the vector sum.*

P 3.1 (commutivity) $|\xi\rangle \oplus |\eta\rangle = |\eta\rangle \oplus |\xi\rangle$.

P 3.2 (associativity) $|\xi\rangle \oplus (|\eta\rangle \oplus |\gamma\rangle) = |(\xi\rangle \oplus |\eta\rangle) \oplus |\gamma\rangle$.

P 3.3 (origin) *There exists within* $\mathcal{V}$ *an element* $\varnothing$ *such that* $|\xi\rangle \oplus \varnothing = |\xi\rangle$.

P 3.4 (inverse) *For every* $|\xi\rangle$ *there exists a unique* $|\xi\rangle$ *in* $\mathcal{V}$ *such that* $|\xi\rangle \oplus |\eta\rangle = |0\rangle$. *Obviously, this means that* $|\xi\rangle = -|\eta\rangle$ *which in turn implies an inverse for each vector.*

Postulate 4 (Scalar Multiplication) *To every pair* $a \in \mathcal{F}$ *and* $\xi \in \mathcal{V}$ *there is a vector which can be formed by the scalar multiplication,* $a \cdot |\xi\rangle$ *which is in* $matchcalV$.

P 4.1 (associativity) $a \cdot (b \cdot |\xi\rangle) = (a \otimes b) \cdot |\xi\rangle$;

P 4.2 (identity) $\mathcal{I} \cdot |\xi\rangle = |\xi\rangle$;

P 4.3 (distributivity–scalar) $(a \oplus b) \cdot |\xi\rangle = a \cdot |\xi\rangle \oplus b \cdot |\xi\rangle$;

P 4.4 (distributivity–vector) $a \cdot (|\xi\rangle \oplus |\eta\rangle) = a \cdot |\xi\rangle \oplus a \cdot |\eta\rangle$.

Definition 4 (Linear Vector Space) *Postulates 3 and 4 specify a Linear Vector Space.*

this is the end?

Postulate 5 (Scalar Product) *A vector space is Unitary if there is an additional product operation defined called the Scalar product,* $\langle\xi|\eta\rangle$ *which is a scalar-valued operation.*

Additional properties which may be possessed by elements of a Unitary Linear Vector Space include:

P 5.1 (Unitarity) $\langle\xi|\eta\rangle = \langle\eta|\xi\rangle^\star$;

P 5.2 (Distributivity) $\langle\xi|\eta \oplus \gamma\rangle = \langle\xi|\eta\rangle \oplus \langle\xi|\gamma\rangle$*;*

P 5.3 (Positive Norm) $\langle\xi|a\eta\rangle = \langle\xi|\eta\rangle a$*;*

P 5.4 (Distributivity) $\langle\xi|\xi\rangle \geq 0$ *and* $\langle\xi|\xi\rangle = 0$ *only if* $|\xi\rangle = 0$

P 5.5 (Linearity) *For* $|\alpha\rangle = |\eta\rangle\, a \oplus |\gamma\rangle\, b$*,* $\langle\xi|\alpha\rangle = \langle\xi|\eta\rangle a \oplus \langle\xi|\gamma\rangle b$ *which is a Linearity condition and* $\langle\alpha|\xi\rangle = \langle\eta|\xi\rangle a^\star \oplus \langle\gamma|\xi\rangle b^\star$*, which is an Antilinear condition.*

Grassmann had anticipated much in the 1840's.

Definition 5 (Outer Product) $[\xi\eta] = -[\eta\xi]$*, it is a non–commutative multiplication operation called by Grassmann the Outer Product.*

Definition 6 (Dual Space) *Having both a linear and an antilinear scalar product leads to the notion of an additional vector space,* $\tilde{\mathcal{V}}$*, called the vector space which is Dual to* $\mathcal{V}$*. If* $|\eta\rangle \in \mathcal{V}$*, then* $\langle\eta| \in \tilde{\mathcal{V}}$*.*

Definition 7 (Vector Components) *Typically, vectors are pictured as an ordered list, and ntuple of complex scalars. That is, the components of vectors belonging to* $\mathcal{V}$*,* $|\xi\rangle$*, can be written as* $\{\xi^1, \xi^2, \ldots \xi^n\}$*. We say that the vector is d dimensional. Components of vectors belonging to the dual space,* $\tilde{\mathcal{V}}$*, can be written as* $\langle\xi| = \{\xi^{1\times}, \xi^{2\times}, \ldots\}$*. More compactly, these sets can be written* $\{\xi^{i\times}\}$ *and* $|\eta\rangle = \{\eta^i\}$*.*

1. Scalar multiplication in component form is represented $|\xi\rangle\, a = \{a\xi^1, a\xi^2, \ldots a\xi d\} = \{a\xi^i\}$.

2. The scalar sum, in component form is $|\alpha\rangle \oplus |\beta\rangle = \{\alpha^1 \oplus \beta^1, \alpha^2 \oplus \beta^2, \oplus \ldots \oplus \alpha^d \oplus \beta^d\}$.

3. The scalar product in component form is represented $\langle\alpha|\beta\rangle = \sum_{i=1}^d \alpha^{i\times}\beta^i$.

Definition 8 (Norm) *The norm of a vector is* $\langle\alpha|\alpha\rangle \equiv |\alpha|^2 = \sum_{i=1}^d |\alpha^i|^2$*. Here,* $|\alpha^i|^2 \equiv \alpha^{i\times}\alpha^i$ *and* $\alpha_i^\dagger \equiv \alpha^{i\times}$*. Hence, the scalar product can be written as* $\langle\xi|\eta\rangle = \sum_{i=1}^d \xi_i^\dagger \eta^i$*.*

Definition 9 (Length of a Vector) *The length of a vector can be written* $||\xi|| \equiv +\sqrt{\langle\xi|\xi\rangle}$*.*

Definition 10 (Einstein Summation Convention) *The Einstein summation convention presumes that indices are summed for identical contravariant and covariant indices. In this notation, the scalar product would be written* $\langle\xi|\eta\rangle = \xi_i^\dagger \eta^i$*.*

Definition 11 (Swartz Inequality) *The Swartz Inequality states* $|\langle\xi|\eta\rangle| \leq |\xi||\eta|$*.*

Definition 12 (Orthogonality) *If $\langle \xi | \eta \rangle = 0$, then these vectors are orthogonal.*

Definition 13 (Linear Combination) *Let $|\alpha\rangle$, $|\beta\rangle$, $|\gamma\rangle$ ... $|\zeta\rangle$, be a set of vectors and let $a_i \in \mathcal{C}$, then the vector $\alpha_0 = |\alpha\rangle a_1 \oplus |\beta\rangle a_2 \oplus |\gamma\rangle a_3 \ldots \oplus |\zeta\rangle a_d$ is a linear combination. The a_i are the coefficients.*

Definition 14 (Linear Independence) *A set of n vectors, $\xi_{[i]}$ are linearly independent if $\sum_{i=1}^{n} \left|\xi_{[i]}\right\rangle a^i = 0$ is not true for values of a^i other than $a^i = 0$. This implies that none of the vectors can be written in terms of the others.*

Definition 15 (span) *A space is said to be spanned by a set of vectors if every vector in that space can be represented by a linear combination.*

Without proof;

1. For a d-dimensional vector space, $\mathcal{V}$, there are no more than d linearly independent vectors spanning $\mathcal{V}$.
2. The vectors belonging to a set of mutually orthogonal vectors form a linearly independent set.
3. It is always possible to form a mutually orthogonal set of vectors from a set which is linearly indendent.
4. No particular linearly independent set of vectors is unique.

Postulate 6 (bases) *Let $|\eta_1\rangle, |\eta_2\rangle, \ldots |\eta_d\rangle$ be a set of linearly independent vectors in $\mathcal{V}_d$ and $|\xi\rangle$ has a unique representation as a linear combination of the $\left|\eta_{(i)}\right\rangle$. If true, then the $\left|\eta_{(i)}\right\rangle$ are bases and span the space, $\mathcal{V}$. (Notice that the index (i) counts the basis vector, not the component of a vector.)*

Postulate 7 (Completeness) *The completeness of a basis set is expressed in the statement*

$$\frac{\sum_{i=1}^{d} \eta^n_{(i)} \eta^{(i)}_m}{\langle \eta^{(i)} | \eta_{(i)} \rangle} = \delta^n_m$$

The indices n, m count the components (the numberator is an equation of components of the vector $|\eta\rangle$), while the index i counts the basis vectors. Sometimes this is called the statement of Closure.

Definition 16 (Basis expansion) *Any basis vector can be expanded in terms of another,*

$$\begin{aligned} \left|\eta_{(i)}\right\rangle &= \left|\gamma_{(j)}\right\rangle \eta^j_{(i)} \\ &= \left|\gamma_{(j)}\right\rangle \mathrm{A}^j{}_i \end{aligned}$$

Likewise,

$$\left|\gamma_{(i)}\right\rangle = \left|\eta_{(k)}\right\rangle \mathrm{B}^{k}{}_{j}$$

These equations require that $\mathrm{B}^{k}{}_{j}\mathrm{A}^{j}{}_{i} = \delta^{k}_{i}$*, which demonstrates that matrix multiplication is the means by which bases transform. Here, the top, first index is the row and the bottom, second index labels the column. In this notation, we must have* $\mathrm{M}^{\mathrm{T}\,j}_{i} = \mathrm{M}^{j}{}_{i}$*. The dual bases transform similarly:* $\left\langle \eta^{(i)}\right| = \sum_j (\mathrm{B}^{j}{}_{i})^{\times} \left\langle \gamma^{(i)}_{j}\right|$*. This requires that*

$$(\mathrm{B}^{j}{}_{i})^{\times} \equiv B^{\times j}_{i} \equiv \mathrm{B}^{\dagger i}{}_{j}$$

. in order for matrix multiplication to be properly defined.

Postulate 8 (Direct Addition Space) *Two vector spaces can be combined as Direct Sums,* $\mathcal{V}_{n+m} = \mathcal{V}^a_n \oplus \mathcal{V}^b_m$ *with the following properties:*

P 8.1 *The vectors of the spaces add:* $|\xi\rangle = |\eta(a)\rangle \oplus |\eta(b)\rangle$ *and the components are dual-valued,* $(\eta(a);\eta(b))$*.*

P 8.2 *Components combine:*
$(\eta(a);\eta(b)) \oplus (\gamma(a);\gamma(b)) = (\eta(a) \oplus \gamma(a);\eta(b) \oplus \gamma(b))$

Postulate 9 (Direct Product Space) *Two vector spaces can be combined as Direct Product,* $\mathcal{V}_{n+m} = \mathcal{V}^a_n \otimes \mathcal{V}^b_m$ *with the following properties:*

P 9.1 *The vectors of the spaces combine* $|\xi\rangle = |\eta(a)\rangle \otimes |\eta(b)\rangle$*.*

P 9.2 *Multiplication by a scalar:*

$$\begin{aligned} (|\eta(a)\rangle\, c) \otimes |\eta(b)\rangle &= c(|\eta(a)\rangle \otimes |\eta(b)\rangle) \text{ and} \\ |\eta(a)\rangle \otimes (|\eta(b)\rangle\, c) &= c(|\eta(a)\rangle \otimes |\eta(b)\rangle \end{aligned}$$

P 9.3 *Components distributively*
$|\eta(a)\rangle \otimes (|\xi(b)\rangle \oplus |\gamma(b)\rangle = |\eta(a)\rangle \otimes |\xi(b)\rangle \oplus |\eta(a)\rangle \otimes |\gamma(b)\rangle$*.*

P 9.4 *Suppose* $\left|\alpha_{(i)}(a)\right\rangle$ *is a basis set in* $\mathcal{V}^a$*and*$\left|\beta_{(j)}\right\rangle$ *in* V^b*. The vector set* $\alpha_{(i)} \otimes |\beta_j\rangle$ *forms a basis of* $\mathcal{V}_{nm}$ *having dimension* $m \cdot n$*.*

P 9.5 *The inner product should be written* $\langle \delta(a)\epsilon(b)|\xi(a)\zeta(b)\rangle = \langle \delta(a)|\xi(a)\rangle\langle \epsilon(b)|\zeta(b)\rangle$*, keeping the spaces separated.*

P 9.6 *If there are orthonormal bases in* $\mathcal{V}^a$ *and* $\mathcal{V}^b$*, then the product bases in* $\mathcal{V}$ *will also be orthonormal.*

Postulate 10 (Compact Function Space) *A compact space is defined by the following: suppose that for a vector $\xi \in \mathcal{V}$ there is a series of vectors $\xi_n(x) = \sum_{i=1}^{n} \eta_{(i)}(x)\xi^i_{[\eta]}$ so that $||\xi(x) - \xi_n|| < \epsilon$ for $n > N(\epsilon)$. If $N = f(\epsilon)$ for all x, then the convergence is "uniform". If $N = f(\epsilon, x)$, then the convergence is "pointwise".*

Definition 17 (Properties of Function Spaces) *Suppose have functions $f(x)$ and $g(x)$ defined over the domain $a \leq x \leq b$ for the continuous variable x, then*

1. *They can be combined into a third function, $h(x) = f(x) + g(x)$.*
2. *A scalar product can be defined as $\langle f|g\rangle = \int_a^b f^\star(x)g(x)dx$.*

Definition 18 (Spectral Decomposition) *Any function which is a vector in a compact vector space can be written in terms of a basis set of functions as a Spectral Expansion. $\xi(x) = \sum_{i=1}^{\infty} \eta_{(i)}(x)\xi^i_{[\eta]}(x)$. The function is defined over the domain $[a, b]$. The basis set is denumerably infinite. The expansion coefficients are $\xi^i_{[\eta]} = \langle \eta_{(i)}(x)|\xi(x)\rangle$. The above condition on convergence is necessary for a spectral decomposition to be defined. The basis functions must be in the set of square-integrable functions, $\mathbb{L}^2(a, b)$*

Postulate 11 (Completeness) *The statement of completeness for functions in a compact vector space is $\lim_{n\to\infty} \int_a^b ||\xi(x) - \sum_{i=1}^{n} \eta(i)\xi^i_{[\eta]}||dx = 0$*

Definition 19 (Hilbert Space) *Vectors which satisfy the requirements of Postulates III. IV, V, X, and XI are members of a Hilbert Space.*

III. The result of a given experiment cannot be predicted precisely. Only the average values can be calculated.
Corollary. The measurement of a quantity carries a system into an eigenstate of the observed quantity.

Definition 20 (State Vector Transformation) *State vectors can be transformed into one another in two ways:*

1. *through dynamics, continuously*
2. *through a measurement, discontinuously*

The mathematics of this is a mapping, $|\xi\rangle \to |\alpha\rangle$. This is presumed to be described by the application of a suitable operator, so $M|\xi\rangle = |\alpha\rangle$

Postulate 12 (Linear Operators) *An operator, M is linear if*

1. *$M(|\alpha\rangle \oplus |\beta\rangle) = M|\alpha\rangle \oplus M|\beta\rangle$*
2. *For $a \in \mathcal{C}$, $M|a\xi\rangle = M|\xi\rangle a$*
3. *It is not necessarily true that $AB|\xi\rangle = BA|\xi\rangle$. Some operators my not commute, that is $[A, b] \neq 0$ may be true according to the physics.*

Postulate 13 (Hermiticity) *An operator is self-adjoint, or Hermitian, if* $M = M^\dagger$ *where according to our notation,*

$$\begin{aligned}
(M^\dagger)^i_j &= (M^j_i)^\times \\
\langle \nu^i | M^\dagger | \eta_j \rangle &= \langle \eta^j | M | \eta_i \rangle^\times
\end{aligned}$$

The operator is Unitary if $MM^\dagger = M^\dagger M = 1$.

Postulate 14 (Eigenvectors and Eigenvalues) *If the following relationship exists,*

$$\begin{aligned}
A \left|\alpha_{(i)}\right\rangle &= \left|\alpha_{(j)}\right\rangle \delta^j_i a(i) \\
&= \left|\alpha_{(i)}\right\rangle a(i)
\end{aligned}$$

then the abstract $|\alpha\rangle$ *are eigenstates of the operator* A *having eigenvalues* $a(i)$. *The label* (i) *for the* a *is not summed.*

(Still work in progress)

IV. **For every quantum mechanical system there exists a family of linear operators defined on the Hilbert Space which describes the evolution of the system between two times.**
Corollary. Unitary transformations are cool and the basic tool of practical quantum mechanics.