Problem Set #2

PHY 853, Fall Semester, 1999

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These problems are due on November 29, 1999 5:00 p.m. in my mailbox.

Problem 5 As was done in class, show that the two dimensional matrix representation for the c element of D_3 is:

$$\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

Problem 6 For D_3, show that $f(x,y) = xy$ is a basis function in the 2-dimensional representation.

Problem 7 Remember that the projection operator for an operation on a function F is,

$$P^{(\ell)}F = \frac{d_\ell}{g} \sum_{\substack{k=group \\ element}}^{order} \chi^{*(\ell)}(\gamma_k)\Gamma(\gamma_k)F$$

and the transformation operator is

$$P_j^{(\ell)i}F = \frac{d_\ell}{g} \sum_{\substack{k=group \\ element}}^{order} \Gamma^{(\ell)}(\gamma_k)_j^{*i}\Gamma(\gamma_k)F.$$

(a) Show that the function $F = x^2$ decomposes into a function which transforms like the $\Gamma^{(1)}$ IRR of D_3 and the two functions which transform among themselves as $\Gamma^{(2)}$ IRR of D_3. The former is the single function $\frac{1}{2}(x^2 + y^2)$ and the latter functions are xy and $\frac{1}{2}(x^2 - y^2)$.

(b) Show, for the b element of D_3, that the latter two functions do indeed transform among themselves.

Problem 8 Refer to the covering group of a square which you worked out in the first problem set.

(a) What does the part of the character table look like for the 2-d represenation?

(b) From the consideration of the elements of the 2d IRR on the function $x^2 - y^2$, one can actually construct a 1d representation. Do that.

(c) How many 1d IRR will there be? (Use the results of your last problem set.)

(d) Using the above plus Orthogonality, construct the complete character table for D_4.

Problem 9 The covering group for the NH_3 molecule is the group C_{3v}. The character table is:

C_{3v}	E	$2C_3$	$3\sigma_v$	
A_1	1	1	1	$z; x^2+y^2; z^2$
E	2	-1	0	$x; y; (x^2-y^2); xy; yz; I_x; I_y$
A_2	1	1	-1	I_z

Show that an electric dipole perturbation will result in transitions between the $E \to A_1$ and $E \to A_2$ levels.

Problem 10 In the Kronecker decomposition for SU(2), we have for the combination of three states:

$$2 \otimes 2 \otimes 2 = 4 \oplus 2 \oplus 2.$$

For the quartet, find all four states in terms of the irreducible SU(2) tensors, $\xi^i \eta^j \gamma^k$.

Problem 11 Complete the formalism to show that that the algebra of SU(2) contains the term

$$[X_1, X_2] = -2X_3.$$

Remember that the infinitesimal transformation on the vectors is

$$\delta\xi^i = \eta^i_j \xi^j = U^i_\sigma \delta\alpha^\sigma,$$

that the infinitesimal transformation matrix is

$$\eta = \begin{pmatrix} i\alpha^1 & \alpha^2 + i\alpha^3 \\ -\alpha^2 + i\alpha^3 & -i\alpha^1 \end{pmatrix},$$

and that the generators are defined as

$$X_\sigma = U^i_\sigma \frac{\partial}{\partial \xi^i}.$$

Problem 12 Show that the asymmetric (singlet) state

$$|\eta_0\rangle = \frac{1}{\sqrt{2}} \left(\left|\frac{1}{2}\right\rangle_1 \otimes \left|-\frac{1}{2}\right\rangle_2 - \left|-\frac{1}{2}\right\rangle_1 \otimes \left|\frac{1}{2}\right\rangle_2 \right)$$

is invariant under $\mathsf{SU}(2)$ rotations. Hint, remember that a rotation here implies

$$|\eta_0\prime\rangle = \exp\left[\frac{i}{2}\boldsymbol{\sigma}_1 \cdot \boldsymbol{\theta}\right] \exp\left[\frac{i}{2}\boldsymbol{\sigma}_2 \cdot \boldsymbol{\theta}\right] |\eta_0\rangle \,,$$

where the subscripts emphasize the fact that the operators seek out vectors in their own spaces.