Problem Set #2

PHY 853, Fall Semester, 1999
Chip Brock, brock@chip.pa.msu.edu

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These problems are due on November 29, 1999 5:00 p.m. in my mailbox.

**Problem 5**
As was done in class, show that the two dimensional matrix representation for the \( c \) element of \( D_3 \) is:

\[
\begin{pmatrix}
\frac{1}{2} & \frac{\sqrt{3}}{2} \\
\frac{\sqrt{3}}{2} & -\frac{1}{2}
\end{pmatrix}
\]

**Problem 6**
For \( D_3 \), show that \( f(x, y) = xy \) is a basis function in the 2-dimensional representation.

**Problem 7**
Remember that the projection operator for an operation on a function \( F \) is,

\[
P^{(\ell)} F = \frac{d}{g} \sum_{k=\text{group element}} \chi^{*^{(\ell)}}(\gamma_k) \Gamma(\gamma_k) F
\]

and the transformation operator is

\[
P^{(\ell)i}_j F = \frac{d}{g} \sum_{k=\text{group element}} \Gamma^{(\ell)}(\gamma_k)_j^{*i} \Gamma(\gamma_k) F.
\]

(a) Show that the function \( F = x^2 \) decomposes into a function which transforms like the \( \Gamma^{(1)} \) IRR of \( D_3 \) and the two functions which transform among themselves as \( \Gamma^{(2)} \) IRR of \( D_3 \). The former is the single function \( \frac{1}{2}(x^2 + y^2) \) and the latter functions are \( xy \) and \( \frac{1}{2}(x^2 - y^2) \).

(b) Show, for the \( b \) element of \( D_3 \), that the latter two functions do indeed transform among themselves.

**Problem 8**
Refer to the covering group of a square which you worked out in the first problem set.

(a) What does the part of the character table look like for the 2-d representation?
(b) From the consideration of the elements of the 2d IRR on the function \(x^2 - y^2\), one can actually construct a 1d representation. Do that.

(c) How many 1d IRR will there be? (Use the results of your last problem set.)

(d) Using the above plus Orthogonality, construct the complete character table for \(D_4\).

**Problem 9** The covering group for the NH\(_3\) molecule is the group \(C_{3v}\). The character table is:

<table>
<thead>
<tr>
<th>(C_{3v})</th>
<th>E</th>
<th>2C(_3)</th>
<th>3(\sigma_v)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A_1)</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(E)</td>
<td>2</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>(A_2)</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
</tbody>
</table>

\(z; x^2 + y^2; z^2\)
\(x; y; (x^2 - y^2); xy; yz; I_x; I_y\)
\(I_z\)

Show that an electric dipole perturbation will result in transitions between the \(E \rightarrow A_1\) and \(E \rightarrow A_2\) levels.

**Problem 10** In the Kronecker decomposition for SU(2), we have for the combination of three states:

\(2 \otimes 2 \otimes 2 = 4 \oplus 2 \oplus 2\).

For the quartet, find all four states in terms of the irreducible SU(2) tensors, \(\xi^i\eta^j\gamma^k\).

**Problem 11** Complete the formalism to show that that the algebra of SU(2) contains the term

\([X_1, X_2] = -2X_3\).

Remember that the infinitesimal transformation on the vectors is

\(\delta \xi^i = \eta^j \xi^j = U^i \delta \alpha^\sigma\),

that the infinitesimal transformation matrix is

\(\eta = \begin{pmatrix} i\alpha^1 & \alpha^2 + i\alpha^3 \\ -\alpha^2 + i\alpha^3 & -i\alpha^1 \end{pmatrix}\),

and that the generators are defined as

\(X_\sigma = U^i \frac{\partial}{\partial \xi^i}\).

**Problem 12** Show that the asymmetric (singlet) state

\(|\psi_0\rangle = \frac{1}{\sqrt{2}} \left( |\frac{1}{2}\rangle_1 \otimes |\frac{1}{2}\rangle_2 - |\frac{1}{2}\rangle_1 \otimes |\frac{1}{2}\rangle_2 \right)\)
is invariant under SU(2) rotations. Hint, remember that a rotation here implies

$$|\eta_0\rangle = \exp\left[\frac{i}{2} \sigma_1 \cdot \theta\right] \exp\left[\frac{i}{2} \sigma_2 \cdot \theta\right] |\eta_0\rangle,$$

where the subscripts emphasize the fact that the operators seek out vectors in their own spaces.