Experiment 5

Simple Harmonic Motion

Goals

1. To understand the properties of an oscillating system governed by Hooke's Law.
2. To study the effects of friction on an oscillating system, which leads to damping.
3. To use a non-linear least-squares fitting procedure to characterize an oscillator.

Theoretical Introduction

Simple Harmonic Oscillation (SHO) Consider a system illustrated in the figure below. It consists of a mass *m* suspended from a spring with spring constant *k*.

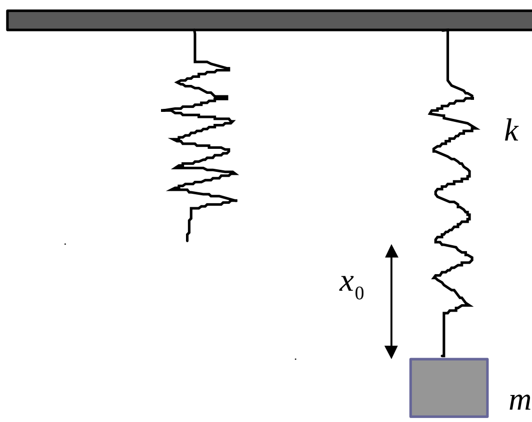

Fig. 1. A mass on a spring in the gravitational field of Earth

If we let the mass hang without moving, then the spring will be stretched to an equilibrium position $x_0 = \frac{mg}{k}$, where *g* is the gravitational constant. This is a direct consequence of Hooke's law, $F = -kx$, where the force retarding the extension of the spring is proportional to the deviation of the spring from its equilibrium position.

If the mass is now displaced from its equilibrium position, the same equation still applies, where *x* is now that displacement. In other words, when such a displacement is made, a restoring force acts to return the mass to its equilibrium position. Upon release, the mass moves toward the equilibrium position, but its inertia causes it to "overshoot" this point. The motion then continues through the equilibrium position and beyond until the restoring force eventually stops the mass and pulls it back toward the equilibrium position. The motion then repeats itself back and forth through the position of equilibrium. Newton's second law states that any unbalanced force results in an acceleration of the mass, proportional to the force. If we apply Newton's second law to the motion of a mass m that is subject to Hooke's law, we get

$$m\frac{d^2x}{dt^2} = -kx\,, \tag{1}$$

which can be written as

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0\,, \qquad \text{where} \quad \omega_0^2 = \frac{k}{m}\,. \tag{2}$$

We have studied the solution of this equation in the previous experiment (Exp3), where the simple pendulum underwent simple harmonic oscillation with angular frequency $\omega = \sqrt{g/L}$, where g is the gravitational constant and L is the effective length. We found that the solution for the simple pendulum was expressed in a sinusoidal form, $\Theta(t) = \Theta_0 \sin(\omega t)$. Therefore, we anticipate that the solution for Eq. (2) should be similar, namely

$$x(t) = A\cos(\omega_0 t + \varphi) + C\,, \tag{3}$$

where A is the amplitude of oscillation, C is an arbitrary constant, φ is a phase constant, and ω_0 is the angular frequency:

$$\omega_0 = \sqrt{\frac{k}{m}}\,. \tag{4}$$

Damped Harmonic Oscillation (DHO) The amplitude of oscillation of the mass gradually decreases over time. This is due to the effect of friction or a drag (resistive) force. We want to see if we can understand its effects. For our example, the effect of friction can be represented as a force proportional to its velocity of the mass. Therefore, Eq. (1) can be modified to read

$$m\frac{d^2x}{dt^2} = -kx - \beta\frac{dx}{dt} \tag{5}$$

where β is a constant of proportionality, called the damping coefficient. The minus sign indicates that the damping force is always opposite to the direction of motion. Rearranging the above equation yields

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x = 0\,, \tag{6}$$

where $\gamma = \dfrac{\beta}{2m}$ and $\omega_0^2 = \dfrac{k}{m}$. The solution to this (second order differential) equation is no longer SHO. If β is not too large, it is a modification of SHO. Also, the frequency of oscillation will be modified by the damping. The solution may be obtained by an educated guess. We postulate that the oscillatory motion is still sinusoidal, but now includes a multiplicative decay function, so that is the amplitude of oscillatory motion decreases as a function of time. So our trial solution is

$$x(t) = Ae^{-\gamma t}\cos(\omega t + \varphi) + C\,, \tag{7}$$

where A is the amplitude, $\omega = \sqrt{\omega_0^2 - \gamma^2}$ is the angular frequency of this system, φ is the phase angle and C is some arbitrary constant. See Appendix A for a more complete derivation. The coordinate $x(t)$, as a function of time t, is shown in Fig. 2.

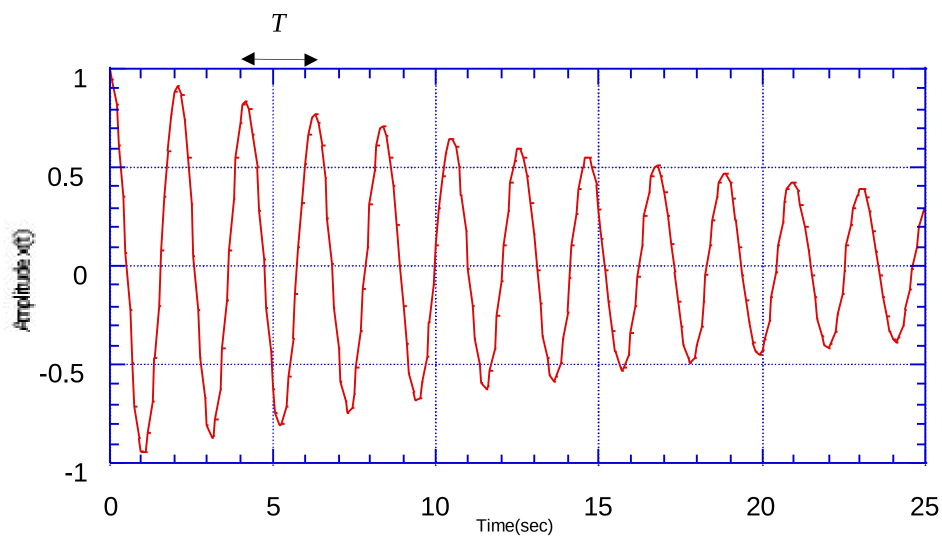

Fig. 2. Damped harmonic oscillator as a function of time. The envelope decay function is exp(-?t). The period T is related to ω_0 by $T = 2\pi/\omega_0$, where $\omega_0 = 2\pi f$.

Questions for the Preliminary Discussion

1. Show by using Eq. (1) that a stationary mass m hanging from a spring with spring constant k (Fig. 1) stretches the spring to a new equilibrium position $x_0 = \dfrac{mg}{k}$.
2. Show by substitution that Eq. (3) is a solution to Eq. (2). Would a solution of the form $x(t) = A\sin(\omega_0 t + \varphi)$ work in Eq. (2)?
3. Confirm by substitution that a solution to Eq. (6) indeed takes the form of Eq. (7).
4. Show that when = 0 (no damping), Eq. (7) reduces to Eq. (3).
5. In Fig. 2, what is the amplitude of the oscillation at $t = 10\,\text{sec}$? at $t = 0\,\text{sec}$? Determine the period T, the angular frequency ω and the phase angle φ.
6. If the phase angle f = p/2, redraw the oscillations in Fig. 2.

III. Experimental Procedure

The Dynamic Force Transducer In this experiment we will use a Dynamic Force Transducer (DFT), an electronic device that outputs a voltage proportional to a force applied to it. In **Parts I and II**, we will hook the outputs of the DFT to the input terminal block of a LabView card and use LabView to monitor the applied force. If we use the DFT as the spring support, then the force exerted by the spring is measured. This force is the same as that applied to the hanging mass and is by Hooke's Law proportional to the displacement of the mass from equilibrium. Thus, by monitoring the force in the spring we are, in effect, monitoring the position of the mass.

Part 0: Preliminary measurements

A. **Calibration of the DFT**. There are two knobs on the DFT, offset (zero adjust) and gain (sensitivity). Set the gain knob nearly fully clockwise and *do not touch it again during the experiment*. Attach the spring-hook-mass (20g) combination to the DFT. Then connect the output of the DFT to the digital oscilloscope. With the spring and mass attached at equilibrium, adjust the offset button of the DFT until the output reads zero volt(s) on the digital scope display. Note that zeroing the output at equilibrium is for convenience only.

1. Measure the output (volts) for three different masses added. The recommended order of masses is 50g, 100g, and 150g. Your instructor will show you how to read the output voltage on the digital scope display.
2. For these three measurements make a graph of V (volts) vs. m (grams). What function describes the dependence of V on m?

B. **Static Measurement of the Force Constant**. In this part you confirm Hooke's Law by measuring the displacement vs. mass using a metric tape measure or a meter stick.

1. Measure displacements for three (3) different masses. Assign uncertainties to the measured displacements and masses.
2. Plot displacements (in units of cm) vs. mass (in units of grams) using *K-graph*. Apply a least-squares fit procedure to obtain the spring constant *k* and its uncertainty and calculate the oscillation frequency ω_0 for each of these masses. Show the calculation in your notebook. What are the units of *k*?

C. **Simple harmonic Oscillations**. In this part of the experiment you will use the hand timer to measure the angular frequency of oscillation ω_0. You will perform this for three different masses.

1. Attach your first mass to the spring and set it into oscillation. Pick an appropriate point of reference for counting the number of complete cycles of the oscillations, then start your timer for ten (10) complete cycles. Record the time (in seconds) in your notebook.
2. Repeat step 1 for your second and third mass.
3. To obtain ω_0, you note that $\omega_0 = 2\pi f$, where f is measured in cycles per second. Using *K-graph*, plot your data of ω_0 vs. m on both a logarithmic plot and a linear

plot. Do you observe the functional dependence predicted by Eq. (4)? Compare these ω_0 with those found in **Part 0: B**.

Part I: Oscillating Spring

We will next consider the case of a mass attached to a spring as shown in Fig. 1. If the spring is now set into motion, stretching or compressing the spring acts to retard its deviation from the equilibrium position and we get back Eq. (2) for the differential equation describing its motion, where x is now the amount the mass deviates from its equilibrium position. The solution is once again $x(t) = A\cos(\omega_0 t) + C$ with the angular frequency $\omega_0 = \sqrt{k/m}$.

1. Attach the spring-hook-mass (100g) combination to the DFT. With the help of the digital scope, adjust the DFT offset. Set the spring-mass system into oscillation and observe the pattern on the scope display and record it in your notebook.
2. Measure the frequency f and the amplitude A on the digital scope using the built in buttons. Calculate the angular frequency ω0 from f and record it in your notebook.
3. Disconnect the output of the DFT from the scope and connect it to the data acquisition card on the PC. Open a program called "Force Transducer.vi" in the LabView folder in C:\LabView\Vi.lib\vi_for_phy191\ForceTransducer.vi. The following specifications are needed in order to collect the data successfully. Set **device** = 1; **channels** = 0; **voltage data:** scan# = 0, channel = 1; **scan numbers** = 100, **scan rate** = 5; **input limits** = 2. Set the spring-mass system into oscillation and press **Ctrl R** to begin collecting the data. (The scanning process will take 20 seconds.) When it is done, the program will ask you to save the data; save it under the folder Phy191 in your designated section folder. (Or you could save it on your floppy disk.) **Caution:** When exiting the program, it will ask you if you want to save the current setting. Choose **NO.**
4. Use *K-graph* to retrieve the data file. When inputting your data to *K-graph*, the following specifications are required: **Delimite**r = Tab; **Number** = 1, **Line Skipped** = 0, **Options/Read Title** (No check). Your file contains two columns: the first column is the time, the second column is the output voltage.
5. Plot voltage vs. time and use the general curve-fit editor to perform a non-linear least-squares fit procedure using the function given in Eq. (3). KaleidaGraphically your function given in Eq. (3) is m1+m2*cos(m3*m0 + m4); m3 and m4 are in radians. Note: This non-linear least-squares fit function will not converge unless your initial parameters (m1…m4) are realistic. Show in your notebook how you estimated these parameters. How do the values compare to those output by *K-graph*?

Part II: The Effect of Friction

In this part of the experiment, we will investigate the behavior of the spring oscillations under the effect of friction.

1. Attach the friction umbrella to your spring-hook-mass (150g) combination and once again observe the spring tension on the digital scope. Adjust the **offset** on the DFT if necessary. Set the system into oscillation and observe the oscillation pattern on your digital scope. You should observe a pattern similar to that in Fig. 2, which shows the sinusoidal oscillation modulation by the exponential damping factor.
2. Measure the frequency f on the digital scope using the built in buttons. Calculate the angular frequency ω and the period T from f and record these values in your notebook.
3. With the friction umbrella attached to your spring-mass system as in step 1, repeat steps 3 and 4 of **Part I**. Fit the data to the function given by Eq. (6): m1+ m2*exp(-m3*m0)*cos(m4*m0 + m5). Show how your m1…m5 parameters are estimated. How do these values compare to those given by *K-graph*?
4. Repeat the experiment with a 100g mass.

Questions to be Discussed

1. Define and explain briefly the meaning of the terms (a) restoring force, (b) free oscillation, (c) simple harmonic motion, (d) phase angle and (e) natural frequency.
2. In **Part 0**: **B** we calculated the spring constant k from the slope of a plot of displacement vs. mass. Using the (same) available data in this **Part,** give an alternative method for making a plot that yields the spring constant k directly from the slope. Hint: What kind of plot gives a slope that is the spring constant k?
3. In this question, you are asked to estimate the decay factor directly from a plot of voltage vs. time. Draw a smooth curve connecting the decay peaks on your plot in **Part II**. The envelope decays as exp(-?t). When t = $1/\gamma$ the amplitude $x(t)$ has decreased by a factor of $1/e$, or about 0.368 times the initial value A. Beginning from any point on the time-axis, determine the length of time required for the amplitude $x(t)$ to decrease by $1/e$ i.e. $1/\gamma$. Solve for γ and equate it to β/2m to determine β. What are the units of β? Compare γ with that given by *K-graph*.
4. Now, compare the angular frequency of free oscillation ω_0 (found in **Part I**) with the damped oscillator frequency ω. Discuss your comparison of ω for the damped oscillator with ω_0 for free oscillator. Use data with the same value of m (100g). Show that $\omega = \sqrt{\omega_0^2 - \gamma^2} = \sqrt{k/m - (\beta/2m)^2}$.
5. Compare γ obtained **Part II** for different masses. Does γ depend on the mass of the system? For a mass of 200g in this system, calculate γ relative to the system with mass of 100g and 150g?

Appendix A

The solution Eq. (7) for the damped harmonic oscillator in Eq. (6) can be found as follows.

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x = 0\,. \tag{A1}$$

The general solution is

$$x(t) = Ae^{Qt}\,, \tag{A2}$$

where Q is a factor to be determined. Then

$$\frac{d}{dt}x(t) = QAe^{Qt} = Q\cdot x(t) \qquad \text{and} \qquad \frac{d^2}{dt^2}x(t) = Q^2Ae^{Qt} = Q^2\cdot x(t)$$

Substitution of these two in (A1) yields

$$(Q^2 + 2\gamma Q + \omega_0^2)x(t) = 0\,, \tag{A3}$$

which has a solution of the form

$$Q = \frac{-2\gamma \pm \sqrt{4\gamma^2 - 4\omega_0^2}}{2} = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}\,. \tag{A4}$$

For our case, which involves a weakly damped oscillation, $\omega_0 >> \gamma$. Therefore, our solution (A4) becomes

$$Q = -\gamma \pm i\sqrt{\omega_0^2 - \gamma^2} = -\gamma \pm i\omega, \quad \text{with } \omega = \sqrt{\omega_0^2 - \gamma^2} \text{ and } \omega_0^2 = k/m \tag{A5}$$

Our general solution Eq. (A2) now becomes

$$x(t) = A_1e^{-\gamma t + i\omega t} + A_2e^{-\gamma t - i\omega t}\,. \tag{A6}$$

Set $A_1 = \frac{1}{2}Ae^{i\varphi}$ and $A_2 = \frac{1}{2}Ae^{-i\varphi}$, where φ is a phase factor. Then

$$x(t) = \frac{A}{2}e^{-\gamma t}e^{i(\omega t+\varphi)} + \frac{A}{2}e^{-\gamma t}e^{-i(\omega t+\varphi)} = Ae^{-\gamma t}\,\frac{1}{2}\left(e^{i(\omega t+\varphi)} + e^{-i(\omega t+\varphi)}\right). \tag{A7}$$

But $\cos(\theta) = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$. Thus,

$$x(t) = Ae^{-\gamma t}\,\frac{1}{2}\left(e^{i(\omega t+\varphi)} + e^{-i(\omega t+\varphi)}\right) = Ae^{-\gamma t}\cos(\omega t + \varphi) + C\,, \tag{A8}$$

which is the solution Eq. (7).

Oscilloscope hints

An oscilloscope (scope) is a device to perform real-time visualization of voltages in an electric circuit. Roughly speaking, you can think about the image on the screen of the scope as the dynamic graph showing the dependence of the measured voltage on time. If the voltage changes, the picture on the screen changes as well. The time is shown on the x-axis, the voltage on the y-axis. Just as in *KGraph*, you can choose the best presentation of the graph by adjusting various scope controls. If you know how many seconds (volts) correspond to one tick on the grid of the screen, you can read the period and the magnitude of the signal directly off the screen.

Setup of the oscilloscope

Your scope can simultaneously measure two voltages; however in this lab we will study only one input signal (supplied through channel A). Both channels A and B have similar sets of controls. During the lab, pay attention that you only work with the controls for channel A, not channel B. The current settings for channel A (B) are shown in the upper part of the digital display to the right from the screen. By pressing the "A/B" button make sure that the shown settings correspond to channel A.

Connect a BNC cable from the transducer to the channel A input. Turn on the scope. Press "AC/DC" button until the digital display shows "DC". Also, turn the "DIGITAL MEMORY" on. Choose the time base "TB" (the value of the 1 cm grid of the time scale) to be equal to "0.5 s", and the voltage base "V – mV" to be equal to "0.1 V". During the lab, you may want to adjust these settings to get the best presentation of the graph.

Press "GND". After this, the scope input will be connected to ground, so that the input voltage is O V. Turn the Y-pos and X-pos knobs until you position the green line of the signal exactly in the middle of the screen. After that, do not touch the X-pos and Y-pos buttons for the rest of your measurements. This way, you will ensure the correct calibration of the picture offsets inside the scope (then you will only need to worry about the voltage offset inside the force transducer which you will adjust with the help of "GAIN" on the box of the transducer).

Press "GND" again. Now you will see the signal coming from the transducer. For non-zero mass, it will be vertically shifted with respect to the center of the screen. For static stretching of the string, you will measure this displacement of the signal to find the dependence between the applied mass, elongation of the string and the voltage. During the study of oscillations, you will have to adjust the "GAIN" on the force transducer until the signal line is again in the middle. Thus you will make sure that the shown voltage corresponds to the displacement of the mass from the equilibrium point, and not from some other point of the system.

Several other useful buttons: you may use the knobs to the left of the screen to adjust the focus and the brightness of the line. The button "LOCK" will allow you to lock (freeze) the image, which makes the measurements more convenient.