Summary of wavefunctions and operators (Chap. 3)

1. The x-space wavefunction

For a free particle,

$$\psi(x,t) = \int A(k) e^{i(kx-\omega t)} \frac{dk}{\sqrt{2\pi}}$$

where $\omega = \dfrac{\hbar k^2}{2m}$.

The momentum operator is $p_{\rm op} = \dfrac{\hbar}{i}\dfrac{\partial}{\partial x}$.

Note that

$$p_{\rm op} e^{ikx} = \hbar k e^{ikx};$$

thus a harmonic wave has a definite wavelength, given by

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{p/\hbar} = \frac{h}{p};$$

this is the de Broglie relation.

2. Expectation values

Define

$$\begin{aligned}
\langle x \rangle &= \int_{-\infty}^{\infty} \psi^*(x,t) x \psi(x,t) dx \\
\langle x^2 \rangle &= \int_{-\infty}^{\infty} \psi^*(x,t) x^2 \psi(x,t) dx \\
\langle f(x) \rangle &= \int_{-\infty}^{\infty} \psi^*(x,t) f(x) \psi(x,t) dx
\end{aligned}$$

Reason: Because $|\psi(x,t)|^2$ is the probability per unit length, dP/dx. Also note that this interpretation requires the normalization condition $\int |\psi(x,t)|^2 dx = 1$.

Similarly,

$$\begin{aligned}
\langle p \rangle &= \int_{-\infty}^{\infty} \psi^* \frac{\hbar}{i} \frac{\partial \psi}{\partial x} dx \\
\langle p^2 \rangle &= \int_{-\infty}^{\infty} \psi^* (-\hbar^2) \frac{\partial^2 \psi}{\partial x^2} dx
\end{aligned}$$

3. The p-space wave function

We may write the plane wave expansion in the form

$$\psi(x,t) = \int \phi(p) e^{i(px-Et)/\hbar} \frac{dp}{\sqrt{2\pi\hbar}}$$

where $E = p^2/2m$. Then $\phi(p)$ is called the momentum-space wave function. By Fourier's theorem,

$$\phi(p) = \int \psi(x) e^{-ipx/\hbar} \frac{dx}{\sqrt{2\pi\hbar}}.$$

Expectation values involving p may be calculated from $\phi(p)$; e.g.,

$$\langle p \rangle = \int_{-\infty}^{\infty} \phi^*(p) p \phi(p) dp.$$