Exam 3 Solutions

① $H = H_x + H_y$ where $H_x = \frac{p_x^2}{2m} + \frac{1}{2} m\omega^2 x^2$

(a) H_y = similarly

Using separation of variables, $u(x,y) = f(x)\,g(y)$ where

$$-\frac{\hbar^2}{2m} f'' + \frac{1}{2} m\omega^2 x^2 f = \mathcal{E}_x f \quad \text{and } g(y) \text{ similarly}$$

∴ $f(x)$ and $g(y)$ are harmonic oscillator eigenfunctions

$$\epsilon_x = \hbar\omega(n_x + 1/2) \quad \text{and} \quad \epsilon_y = \hbar\omega(n_y + 1/2)$$

$$E = \hbar\omega(n_x + n_y + 1)$$

(b)

energy/$\hbar\omega$	quantum nos.	degeneracy
4	(3,0) (2,1) (1,2) (0,3)	4
3	(2,0) (1,1) (0,2)	3
2	(1,0) and (0,1)	2
1	(0,0)	1
0		

② The radial function is $R(r) = j_0(kr)$ where $E = \frac{\hbar^2 k^2}{2m}$

The boundary condition $R(a) = 0 \implies j_0(ka) = 0$.

Then $ka = n\pi$ where n is an integer, ≥ 1.

Ground state has $n = 1$.

$$E = \frac{\hbar^2}{2m}\left(\frac{\pi}{a}\right)^2 = \frac{(\hbar c)^2}{2\,mc^2}\frac{\pi^2}{a^2}$$

$$= \frac{4\times 10^{-10}\ \mathrm{eV}^2\,\mathrm{cm}^2 \times 10}{2 \times 0.5\times 10^6\ \mathrm{eV} \times 10^{-16}\ \mathrm{cm}^2}$$

$$= 4\times 10\ \mathrm{eV} = \boxed{40\ \mathrm{eV}}.$$

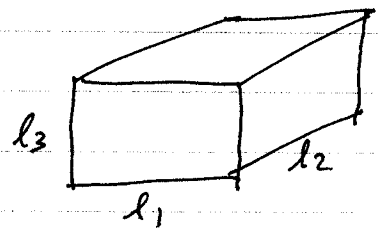
(3) $$N = 2\sum_{\vec{k}} = 2\,\frac{V}{(2\pi)^3}\int_0^{k_F} 4\pi k^2\,dk = \frac{V}{(2\pi)^3}\,\frac{8\pi}{3}\,k_F^3$$

$$\therefore\quad k_F = \left[\frac{(2\pi)^3}{V}\,\frac{3N}{8\pi}\right]^{1/3} = \pi\left(\frac{3n}{\pi}\right)^{1/3}$$

$$E_F = \frac{\hbar^2 k_F^2}{2m} \quad \text{so} \quad T_F = \frac{\hbar^2 k_F^2}{2mk} = \frac{\hbar^2\pi^2}{2mk}\left(\frac{3n}{\pi}\right)^{2/3}$$

$$T_F = 82{,}000\ \text{K}.$$

(4)



The energy is

$$E = \frac{\hbar^2}{2m}\left[\left(\frac{n_1\pi}{l_1}\right)^2 + \left(\frac{n_2\pi}{l_2}\right)^2 + \left(\frac{n_3\pi}{l_3}\right)^2\right]$$

depending on 3 quantum numbers, n_1, n_2, n_3.

The ground state has $n_1 = n_2 = n_3 = 1$.

(a) $$E_a = \frac{\hbar^2\pi^2}{2m}\left[\frac{1}{a^2} + \frac{1}{a^2} + \frac{1}{a^2}\right) = \frac{3\hbar^2\pi^2}{2ma^2}$$

(b) $$E_b = \frac{\hbar^2\pi^2}{2m}\left[\frac{1}{(2a)^2} + \frac{1}{(a/2)^2} + \frac{1}{a^2}\right] = \frac{21\,\hbar^2\pi^2}{8ma^2}$$

$$\frac{1}{4} + 4 + 1 = \frac{21}{4}$$

Note that $E_b > E_a$.

$$\frac{E_b}{E_a} = \frac{21/8}{3/2} = \frac{7}{4}.$$