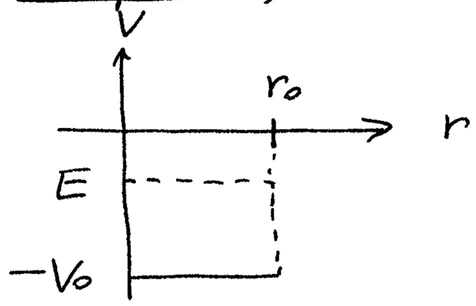
Homework Set 11 (due Nov ~~15~~ 14)

Chapter 10, Problem 2



Deuteron binding:

$l = 0$

$\mu = \frac{m_p m_n}{m_p + m_n} = \frac{1}{2} m_n = \frac{1}{2}(939\ MeV/c^2)$

$E = -2.18\ MeV$

$r_0 = 2.8 \times 10^{-13}\ cm$

What is the depth of the potential, V_0?

The eigenvalue condition is (10-93) with $l=0$.

Note that $j_0(\rho) = \frac{\sin\rho}{\rho}$ and $h_0^{(1)}(\rho) = \frac{e^{i\rho}}{i\rho}$.

Therefore (letting $a = r_0$)

$$\kappa \frac{dj_0/d\rho}{j_0}\Big|_{\rho=\kappa a} = \kappa\left\{\cot\rho - \frac{1}{\rho}\right\}_{\rho=\kappa a} = \kappa\cot\kappa a - \frac{1}{a}$$

and

$$i\alpha\frac{dh_0/d\rho}{h_0}\Big|_{\rho=i\alpha a} = i\alpha\left\{i - \frac{1}{\rho}\right\}_{\rho=i\alpha a} = -\alpha - \frac{1}{a}$$

The eigenvalue condition is

$$\kappa\cot\kappa a = -\alpha$$

or

$$\kappa a\cot\kappa a = -\alpha a \qquad (*)$$

Using (10-90),

$$\alpha a = \sqrt{\frac{2\mu a^2(-E)}{\hbar^2}} = \sqrt{\frac{m_n r_0^2(-E)}{\hbar^2}}$$

$$= \sqrt{\frac{939\ MeV \times (2.8\ fm)^2 \times (2.18\ MeV)}{(197\ MeV\ fm)^2}} = 0.643.$$

Now solve (*) for κa:

$$\kappa a \cot \kappa a = -0.643$$

$$\therefore \kappa a = 1.898$$

Now use (10-90) to determine V_0:

$$\frac{2\mu}{\hbar^2}(E+V_0) = \kappa^2 \Rightarrow V_0 = -E + \frac{\hbar^2}{2\mu a^2}(\kappa a)^2$$

$$V_0 = (2.18 \text{ MeV}) + (5.27 \text{ MeV})(1.898)^2$$

$$\boxed{V_0 = 21.2 \text{ MeV}}$$

(10 - 9) The probability current is

$$\vec{j} = \frac{\hbar}{2i\mu}\left[\psi^* \nabla\psi - (\nabla\psi^*)\psi\right]$$

where we are to consider $\psi(\vec{r}) = C\,\frac{e^{\pm ikr}}{r}\,Y_{\ell m}(\theta,\varphi)$.

The radial component of $\vec{j}$ is the same with $\nabla \to \hat{r}\frac{\partial}{\partial r}$.

Note that

$$\frac{\partial\psi}{\partial r} = C\left(\pm\frac{ik}{r} - \frac{1}{r^2}\right) e^{\pm ikr}\, Y_{\ell m}(\theta,\varphi)$$

Thus

$$j_r = \frac{\hbar}{2i\mu}\Big\{ C^* \frac{e^{\mp ikr}}{r} Y_{\ell m}^* \, C\left(\pm\frac{ik}{r} - \frac{1}{r^2}\right) e^{\pm ikr}\, Y_{\ell m}$$

$$- C^*\left(\mp\frac{ik}{r} - \frac{1}{r^2}\right) e^{\mp ikr}\, Y_{\ell m}^* \, C\,\frac{e^{\pm ikr}}{r}\, Y_{\ell m}\Big\}$$

$$= \frac{\hbar}{2i\mu}\,\frac{|C|^2}{r}\left(\pm\frac{2ik}{r}\right)|Y_{\ell m}|^2$$

$$j_r = \frac{\pm 2\hbar k}{2\mu r^2}\,|C|^2\,|Y_{\ell m}|^2$$

The integrated flux is

$$\Phi = \int \hat{r}\cdot\vec{j}\, d\Omega = \int j_r\, d\Omega = \pm\frac{\hbar k}{\mu r^2}\,|C|^2 \underbrace{\int |Y_{\ell m}|^2 d\Omega}_{=1}$$

$$\boxed{\Phi = \pm\frac{\hbar k}{\mu r^2}\,|C|^2}$$

The flux is outward (i.e. positive) for e^{+ikr}/r

or inward (i.e. negative) for e^{-ikr}/r.

<u>10-11</u> (This problem will turn out to be too difficult to solve completely at this time.)

Start with the radial equation in the form of (10-44), with $l=0$:

$$\frac{d^2u}{dr^2} + \frac{2\mu}{\hbar^2}\left[E - V(r)\right]u = 0 \qquad (1)$$

where $V(r)$ is the Morse potential,

$$V(r) = V_0\left[e^{2(r_0-r)/a} - 2e^{(r_0-r)/a}\right]$$

Let $x = C' e^{-r/a}$, and change the independent variable to x. Note that

$$\frac{du}{dr} = \frac{du}{dx}\frac{dx}{dr} = -\frac{x}{a}\frac{du}{dx}$$

$$\frac{d^2u}{dr^2} = -\frac{1}{a}\frac{d}{dx}\left(x\frac{du}{dx}\right)\frac{dx}{dr} = -\frac{1}{a}\left[\frac{du}{dx} + x\frac{d^2u}{dx^2}\right]\left(\frac{-x}{a}\right)$$

$$= \frac{1}{a^2}\left(x\frac{du}{dx} + x^2\frac{d^2u}{dx^2}\right)$$

Substituting this into (1) we have

$$x^2\frac{d^2u}{dx^2} + x\frac{du}{dx} + \frac{2\mu a^2}{\hbar^2}\left[E - V\right]u = 0$$

Also, let $E = \varepsilon V_0$ (where ε is dimensionless)

and $\lambda = \frac{2\mu a^2}{\hbar^2}V_0$ (λ is dimensionless)

In terms of x, the Morse potential is

$$V(r) = V_0 \left[\left(e^{r_0/a} \frac{x}{c}\right)^2 - 2\left(e^{r_0/a}\frac{x}{c}\right) \right]$$

$$= V_0 \left[x^2 - 2x \right] \quad \text{letting } c \equiv e^{r_0/a} .$$

Thus, the equation for $u(x)$ is

$$\boxed{x^2 \frac{d^2u}{dx^2} + x\frac{du}{dx} + \lambda\left[\varepsilon - x^2 + 2x\right] u = 0} \qquad (2)$$

This equation resembles the Schrodinger radial equation for the hydrogen atom [see eq. (12-4)] and it may be solved by the same techniques as used for the H atom.

We'll complete the solution of (2) in Phys 472.

A plot of $V(r)$ is shown from Mathematica program. $V(r)$ is a model for the potential of 2 atoms in a diatomic molecule. For states near the minimum of $V(r)$, the potential is approximately quadratic so the molecular vibrations are approximately harmonic oscillator states.

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(* Morse potential *)
{V0, r0, a} = {1, 1, 1};
V[r_] := V0 * (Exp[2 (r0 - r) / a] - 2 Exp[(r0 - r) / a])
Vp[r_] := V'[r];
fr = FindRoot[Vp[r] == 0, {r, 1}];
Vmin = V[r /. fr];
Vmax = Max[{N[V[0]], 1}];
Plot[V[r], {r, 0, 5 r0},
  PlotRange → {All, {Vmin, Vmax}}]
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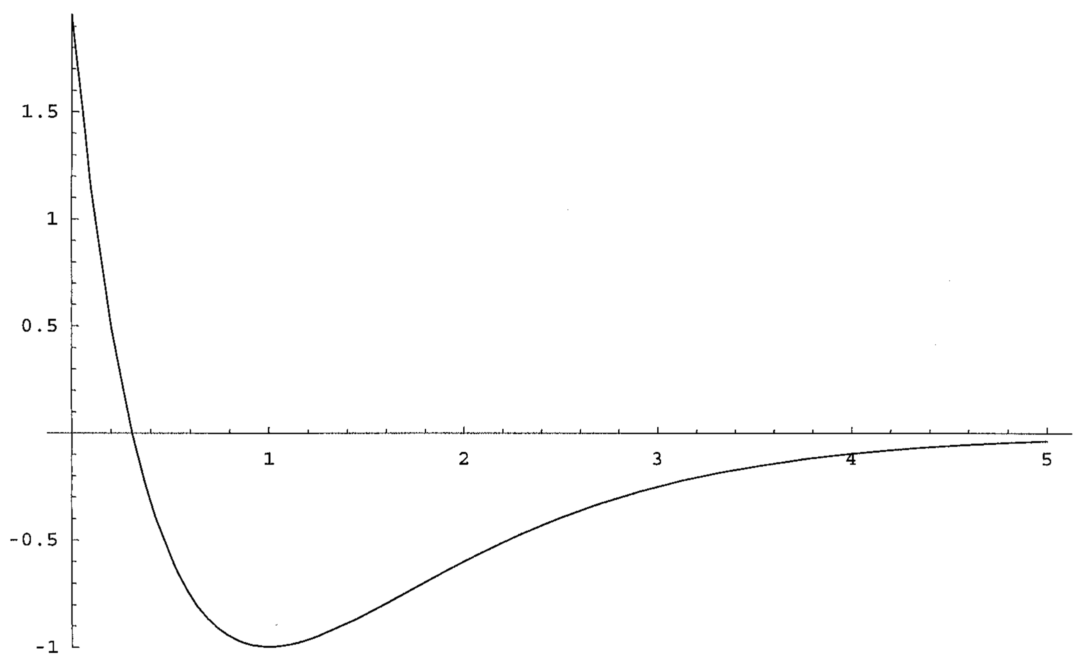

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