Chapter 10, Problem 2

Deuteron binding:

\[ l = 0 \]

\[ \mu = \frac{m_p m_n}{m_p + m_n} = \frac{1}{2} m_n = \frac{1}{2} (939 \text{ MeV}/c^2) \]

\[ E = -2.18 \text{ MeV} \]

\[ r_0 = 2.8 \times 10^{-13} \text{ cm} \]

What is the depth of the potential, \( V_0 \)?

The eigenvalue condition is (10-93) with \( l = 0 \).

Note that \( J_0 (s) = \frac{\sin s}{s} \) and \( l_n (s) = \frac{e^{is}}{is} \).

Therefore (letting \( a = r_0 \))

\[ \kappa \frac{d J_0}{d s} \bigg|_{s = \kappa a} = \kappa \left\{ \cot s - \frac{1}{s} \right\} \bigg|_{s = \kappa a} = \kappa \cot \kappa a - \frac{1}{\kappa a} \]

and

\[ i \kappa \frac{d l_n}{d s} \bigg|_{s = i\omega a} = i\kappa \left\{ -i - \frac{1}{s} \right\} \bigg|_{s = i\omega a} = -\omega - \frac{1}{\kappa a} \]

The eigenvalue condition is

\[ \kappa \cot \kappa a = -\omega \]

or

\[ \kappa a \cot \kappa a = -\omega a \quad (k) \]

Using (10-90),

\[ \omega a = \sqrt{\frac{2 \mu a^2 (\omega)}{h^2}} = \sqrt{\frac{m_n r_0^2 (\omega)}{h^2}} \]

\[ = \sqrt{\frac{939 \text{ MeV} \times (2.8 \text{ fm})^2 \times (2.18 \text{ MeV})}{(197 \text{ MeV fm})^2}} = 0.643. \]
Now solve (4) for $\kappa a$:

$$\kappa a \cot \kappa a = -0.643$$

$$\therefore \kappa a = 1.898.$$

Now we (10-80) to determine $V_0$:

$$\frac{2\mu}{h^2} (E + V_0) = \kappa^2 \Rightarrow V_0 = -E + \frac{b^2}{2\mu a^2} (\kappa a)^2$$

$$V_0 = (2.18 \text{ MeV}) + (5.27 \text{ MeV}) (1.898)^2$$

$$\Rightarrow V_0 = 21.2 \text{ MeV}$$
The probability current is
\[ \vec{J} = \frac{ih}{2i\hbar} \left[ \psi^* \nabla \psi - (\nabla \psi^*) \psi \right] \]
where we are to consider \( \psi(\mathbf{r}) = C \frac{e^{\pm i \mathbf{k} \cdot \mathbf{r}}}{r} Y_m(\theta, \phi) \).

The radial component of \( \vec{J} \) is the same with \( \nabla \rightarrow \frac{\partial}{\partial r} \).

Note that
\[ \frac{\partial \psi}{\partial r} = C \left( \pm \frac{ik}{r} - \frac{1}{r^2} \right) \frac{e^{\pm i \mathbf{k} \cdot \mathbf{r}}}{r} Y_m(\theta, \phi) \]

Thus
\[ J_r = \frac{\hbar}{2i\hbar} \left\{ C^* \frac{e^{i \mathbf{k} \cdot \mathbf{r}}}{r} Y_m^* C \left( \pm \frac{ik}{r} - \frac{1}{r^2} \right) e^{\pm i \mathbf{k} \cdot \mathbf{r}} Y_m ight. \\
- C^* \left( \pm \frac{ik}{r} - \frac{1}{r^2} \right) e^{i \mathbf{k} \cdot \mathbf{r}} Y_m^* C \frac{e^{\pm i \mathbf{k} \cdot \mathbf{r}}}{r} Y_m \right\} \]
\[ = \frac{\hbar}{2i\hbar} \frac{|C|^2}{r} \left( \pm \frac{2ik}{r} \right) |Y_m|^2 \]
\[ J_r = \frac{\pm 2ik}{2\mu r^2} |C|^2 |Y_m|^2 \]

The integrated flux is
\[ \Phi = \int \vec{J} \cdot d\mathbf{A} = \int J_r d\mathbf{A} = \pm \frac{\hbar k}{\mu r^2} |C|^2 \int |Y_m|^2 d\Omega \]
\[ \Phi = \pm \frac{\hbar k}{\mu r^2} |C|^2 \]

The flux is outward (i.e. positive) for \( e^{+ i \mathbf{k} \cdot \mathbf{r}} \)
or inward (i.e. negative) for \( e^{- i \mathbf{k} \cdot \mathbf{r}} \).
10-11 (This problem will turn out to be too difficult to solve completely at this time.)

Start with the radial equation in the form of (10-44), with \( l = 0 \):

\[
\frac{d^2u}{dr^2} + \frac{2l}{r} \left[ E - V(r) \right] u = 0 \tag{1}
\]

where \( V(r) \) is the diverse potential,

\[ V(r) = V_0 \left[ e^{2(r_0 - r)/a} - 2 e^{(r - r)/a} \right] \]

Let \( x = e^{-r/a} \) and change the independent variable to \( x \). Write that

\[ \frac{du}{dr} = \frac{du}{dx} \frac{dx}{dr} = -\frac{x}{a} \frac{du}{dx} \]

\[
\frac{d^2u}{dr^2} = -\frac{1}{a} \frac{d}{dx} \left( x \frac{du}{dx} \right) \frac{dx}{dr} = -\frac{1}{a} \left[ \frac{du}{dx} + x \frac{d^2u}{dx^2} \right] \left( \frac{1}{a} \right) 
\]

\[ = \frac{1}{a^2} \left( x \frac{du}{dx} + x^2 \frac{d^2u}{dx^2} \right) \]

Substituting this into (1) we have

\[ x^2 \frac{d^2u}{dx^2} + x \frac{du}{dx} + \frac{2ma^2}{h^2} \left[ E - V \right] u = 0 \]

Also, let \( \lambda = E V_0 \) (where \( \lambda \) is dimensionless)

and \( \lambda = \frac{2ma^2}{h^2} V_0 \) (\( \lambda \) is dimensionless)
In terms of $x$, the Morse potential is

$$V(r) = V_0 \left[ \left( e^{\alpha r/a} \frac{x}{c} \right)^2 - 2 \left( e^{\alpha r/a} \frac{x}{c} \right) \right]$$

$$= V_0 \left[ x^2 - 2x \right] \text{ with } c = e^{\alpha r/a}.$$  

Thus, the equation for $u(x)$ is

$$x^2 \frac{d^2 u}{dx^2} + x \frac{du}{dx} + 2 \left[ \varepsilon - x^2 + 2x \right] u = 0$$  \(\text{(7)}\)

This equation resembles the Schrödinger radial equation for the hydrogen atom [see eq. (12-4)] and it may be solved by the same techniques as used for the $H$ atom.

We'll complete the solution of (2) in Part 472.

A plot of $V(r)$ is shown from Mathematica program. $V(r)$ is a model for the potential of 2 atoms in a diatomic molecule. For states near the minimum of $V(r)$, the potential is approximately quadratic so the molecular vibrations are harmonic oscillator states.
(* Morse potential *)
{V0, r0, a} = {1, 1, 1};
V[r_] := V0*(Exp[2*(r0 - r)/a] - 2*Exp[(r0 - r)/a])
Vp[r_] := V'[r];
fr = FindRoot[Vp[r] == 0, {r, 1}];
Vmin = V[r /. fr];
Vmax = Max[{-N[V[0]], 1}];
Plot[V[r], {r, 0, 5 r0},
     PlotRange -> {All, {Vmin, Vmax}}]