Homework Set 13 (due Nov 28)

Chapter 11 Exercises 2 3 5 10 11

$\boxed{11-2}$ Spherical harmonics from Equation (11-65)

	in θ, φ	in x, y, z
Y_{00}	$1/\sqrt{4\pi}$	$1/\sqrt{4\pi}$
Y_{11}	$-\sqrt{\frac{3}{8\pi}}\, e^{i\varphi} \sin\theta$	$-\sqrt{\frac{3}{8\pi}}\, \frac{x+iy}{r}$
Y_{10}	$\sqrt{\frac{3}{4\pi}} \cos\theta$	$\sqrt{\frac{3}{4\pi}}\, \frac{z}{r}$
Y_{22}	$\sqrt{\frac{15}{32\pi}}\, e^{2i\varphi} \sin^2\theta$	$\sqrt{\frac{15}{32\pi}}\, \frac{x^2-y^2+2ixy}{r^2}$

Note: $\cos 2\varphi \sin^2\theta + i \sin 2\varphi \sin^2\theta$

$(\cos^2\varphi - \sin^2\varphi) \sin^2\theta + 2i \sin\varphi \cos\varphi \sin^2\theta$

$x^2 - y^2 + 2i\, xy$

Y_{21}	$-\sqrt{\frac{15}{8\pi}}\, e^{i\varphi} \sin\theta \cos\theta$	$-\sqrt{\frac{15}{8\pi}}\, \frac{(x+iy)z}{r^2}$
Y_{20}	$\sqrt{\frac{5}{16\pi}}\, (3\cos^2\theta - 1)$	$\sqrt{\frac{5}{16\pi}}\, \frac{2z^2-x^2-y^2}{r^2}$

$\boxed{11-3}$ Note that $L_\pm = L_x \pm i L_y$ are the raising and lowering operators for m:

$$L_+ |l, m\rangle = c_+(l,m)\, |l, m+1\rangle$$
$$L_- |l, m\rangle = c_-(l,m)\, |l, m-1\rangle$$

where $c_+(l,m)$ and $c_-(l,m)$ are given in (11-48)

$$c_\pm(l,m) = \hbar\sqrt{(l \mp m)(l \pm m + 1)}.$$

$\underline{L_x = \frac{1}{2}(L_+ + L_-)}$

$$\langle l, m_1 | L_x | l, m_2\rangle = \frac{1}{2}\langle l, m_1| \{ c_+(l, m_2)\, |l, m_2+1\rangle + c_-(l,m_2)\, |l, m_2-1\rangle \}$$

$$= \frac{1}{2} c_+(l, m_2)\, \delta(m_1, m_2+1) + \frac{1}{2} c_-(l,m_2)\, \delta(m_1, m_2-1)$$

I've used the fact that $\langle l, m_1 | l, m_2'\rangle = \delta(m_1, m_2')$ (orthonormality)

$\underline{L_y = \frac{1}{2i}(L_+ - L_-)}$

$$\langle l, m_1 | L_y | l, m_2\rangle = \frac{1}{2i}\langle l, m_1| \{ c_+(l, m_2)\, |l, m_2+1\rangle - c_-(l,m_2)\, |l, m_2-1\rangle \}$$

$$= \frac{1}{2i} c_+(l, m_2)\, \delta(m_1, m_2+1) - \frac{1}{2i} c_-(l,m_2)\, \delta(m_1, m_2-1).$$

$\boxed{11\text{-}5}$ The axially symmetric rotator

$$H = \frac{L_x^2 + L_y^2}{2I_1} + \frac{L_z^2}{2I_2} = \frac{L^2}{2I_1} + L_z^2\left(\frac{1}{2I_2} - \frac{1}{2I_1}\right)$$

The eigenstates of H are simultaneous eigenstates of L^2 and L_z: call these state $|\ell, m\rangle$, where

$$L^2|\ell, m\rangle = \hbar^2 \ell(\ell+1)|\ell, m\rangle$$

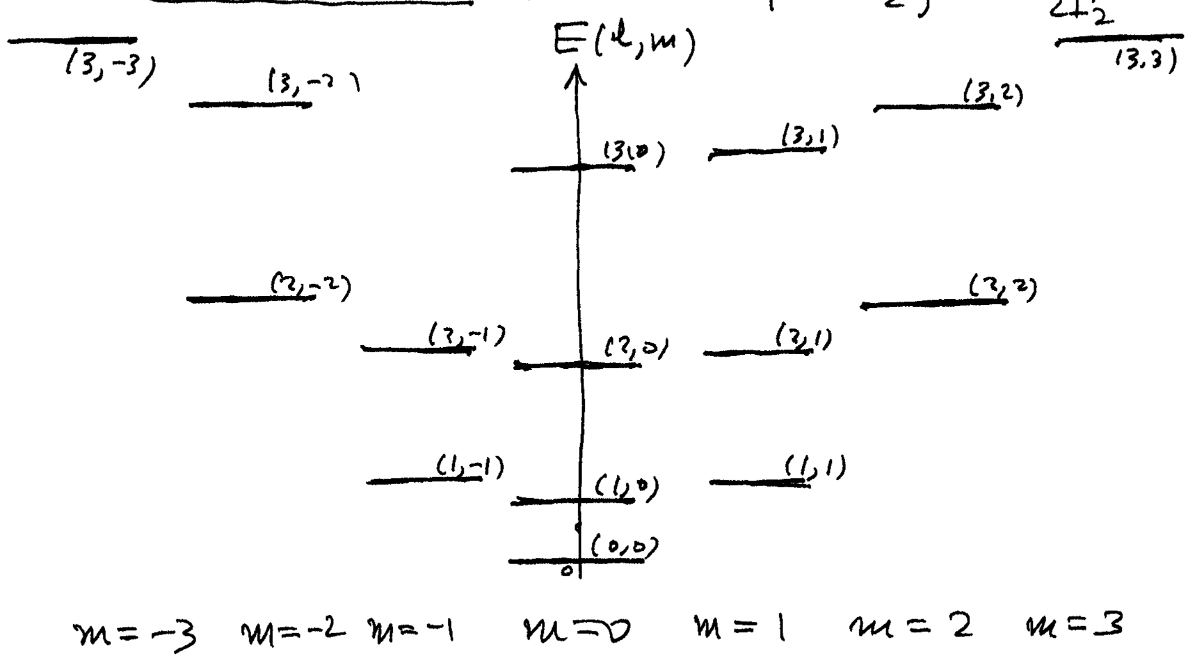
$$L_z|\ell, m\rangle = \hbar m|\ell, m\rangle$$

$$H|\ell, m\rangle = E(\ell, m)|\ell, m\rangle$$

The energy eigenvalues are

$$\boxed{E(\ell, m) = \frac{\hbar^2}{2I_1}\ell(\ell+1) + \hbar^2 m^2\left(\frac{1}{2I_2} - \frac{1}{2I_1}\right).}$$

<u>The spectrum</u> Assume $I_1 > I_2$, so $\frac{1}{2I_2} - \frac{1}{2I_1}$ is positive.



$\boxed{11-10}$ $\psi(x,y,z) = C(xy + yz + zx)e^{-\alpha r^2}$

Expand $\psi(\vec{x})$ in spherical harmonics; i.e., write $\psi(\vec{x}) = \sum_{\ell,m} A_{\ell m} Y_{\ell m}(\theta,\varphi)\, r^2 e^{-\alpha r^2}$. Then the probability that a measurement of angular momentum would yield state with quantum numbers ℓ, m is $\propto |A_{\ell m}|^2$.

Expansion: First, use (11-65) and (11-60) to ~~write~~ identify the relevant spherical harmonics

$$Y_{2,2} = \sqrt{\frac{15}{32\pi}}\, e^{2i\varphi} \sin^2\theta$$

$$Y_{2,-2} = Y_{2,2}^{*} = \sqrt{\frac{15}{32\pi}}\, e^{-2i\varphi} \sin^2\theta$$

$$Y_{2,1} = -\sqrt{\frac{15}{8\pi}}\, e^{i\varphi} \sin\theta\cos\theta$$

$$Y_{2,-1} = -Y_{2,1}^{*} = +\sqrt{\frac{15}{8\pi}}\, e^{-i\varphi} \sin\theta\cos\theta$$

Now express xy, yz, and zx in spherical harmonics

- $$xy = r^2 \sin^2\theta \underbrace{\cos\varphi \sin\varphi}_{\frac{1}{2}\sin 2\varphi = \frac{1}{4i}(e^{2i\varphi} - e^{-2i\varphi})}$$

$$= \frac{r^2}{4i}\left(e^{2i\varphi}\sin^2\theta - e^{-2i\varphi}\sin^2\theta\right)$$

$$= \frac{r^2}{2i}\sqrt{\frac{8\pi}{15}}\,(Y_{2,2} - Y_{2,-2})$$

- $$yz = r^2 \cos\theta \sin\theta \sin\varphi = \frac{r^2}{2i}\sin\theta\cos\theta\,(e^{i\varphi} - e^{-i\varphi})$$

$$= \frac{r^2}{2i}\sqrt{\frac{8\pi}{15}}\,(-Y_{2,1} - Y_{2,-1})$$

- $$zx = r^2 \cos\theta \sin\theta \cos\varphi = \frac{r^2}{2}\sin\theta\cos\theta\,(e^{i\varphi} + e^{-i\varphi})$$

$$= \frac{r^2}{2}\sqrt{\frac{8\pi}{15}}\,(-Y_{2,1} + Y_{2,-1})$$

Thus, substituting back into the formula for $\psi(\vec{x})$,

$$\psi(\vec{x}) = \frac{C}{2}\sqrt{\frac{8\pi}{15}}\, r^2 e^{-\alpha r^2}\Big\{-i\, Y_{2,2} + i\, Y_{2,-2} + (i-1)\, Y_{2,1} + (i+1)\, Y_{2,-1}\Big\}$$

The probability that a measurement of angular momentum would yield $L^2 = \hbar^2 \ell(\ell+1)$ and $L_z = \hbar m$ is proportional to $|\text{coeff. of } Y_{\ell m}|^2$.

Let $P(\ell, m)$ = probability of ℓ, m,

$P(0,0) = 0$ because no term like Y_{00}

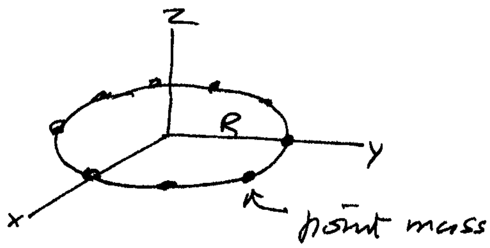
$\sum_{m=-2}^{2} P(2, m) = 1$ because all terms have $\ell = 2$

m	$P(2,m)$	$P(2,m)$
2	a	$1/6$
1	$2a$	$2/6$
0	0	0
-1	$2a$	$2/6$
-2	a	$1/6$

<u>Note</u>

$|i|^2 = 1$, $|-i|^2 = 1$, $|i-1|^2 = 2$, and $|i+1|^2 = 2$.

11-11



N point masses
Total mass = M
Moment of inertia $I = MR^2$.

The energy is $H = \frac{L_z^2}{2I} = \frac{L_z^2}{2MR^2}$

The energy eigenstates are also eigenstates of $L_z = \frac{\hbar}{i}\frac{\partial}{\partial \varphi}$.

Thus $\frac{\hbar}{i}\frac{\partial u}{\partial \varphi} = \hbar m \, u(\phi)$

$$u(\varphi) = \frac{1}{\sqrt{2\pi}} e^{im\varphi}.$$

Rotation about the z axis by angle $\frac{2\pi}{N}$ produces an identical configuration because the masses are identical. Therefore,

$$u\left(\varphi + \frac{2\pi}{N}\right) = u(\varphi)$$

$$e^{im\,2\pi/N} = 1$$

$\frac{2\pi m}{N}$ must be a multiple of 2π

$\boxed{m = N\nu \text{ where } \nu \text{ is an integer.}}$ $(=0, 1, 2, 3 \ldots)$

So the allowed values of m are 0, N, 2N, 3N, ...
The energy is

$$E(m) = \frac{\hbar^2 m^2}{2MR^2} = \frac{\hbar^2 N^2}{2MR^2}\nu^2.$$

The energy of the 1st excited state is $E(1) = \frac{\hbar^2 N^2}{2MR^2}$; this $\to \infty$ as $N \to \infty$

For the "notched cylinder", the allowed values of m are 0, 1, 2, 3, ... The energy of the excited state is then $\hbar^2/2MR^2$, which is ~~constant~~ finite as $N \to \infty$.