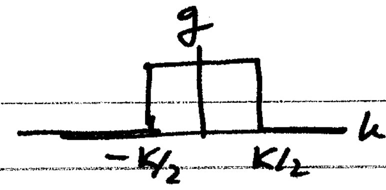
Homework assignment 2 (due Wed, Sep. 12)

Chapter 2



① $g(k) = \begin{cases} N & \text{for } -K/2 \le k \le K/2 \\ 0 & \text{otherwise} \end{cases}$

(a) Form of $f(x)$ $\quad f(x) = \int_{-\infty}^{\infty} g(k)\, e^{ikx}\, dk = N \int_{-K/2}^{K/2} e^{ikx}\, dk$

$$= N \frac{1}{ix}\left(e^{iKx/2} - e^{-iKx/2}\right) = \boxed{\frac{2N}{x} \sin\left(\frac{Kx}{2}\right)}$$

(b) Require $\int_{-\infty}^{\infty} |f(x)|^2 dx = 1 = 4N^2 \int_{-\infty}^{\infty} \frac{\sin^2(Kx/2)}{x^2}\, dx$

$$= 4N^2 \frac{K}{2} \underbrace{\int_{-\infty}^{\infty} \frac{\sin^2 \xi}{\xi^2}\, d\xi}_{=\pi} \quad \text{where } \xi = \frac{Kx}{2}$$

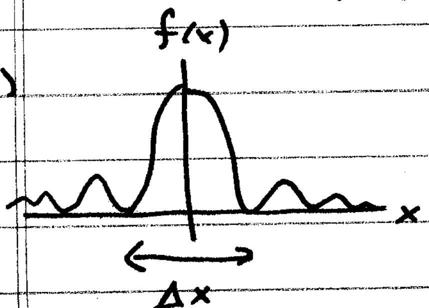
$$dx = \frac{2}{K}\, d\xi$$

So, $2\pi N^2 K = 1 \implies \boxed{N = \frac{1}{\sqrt{2\pi K}}}$

(c) Require $\int_{-\infty}^{\infty} |g(k)|^2 dk = \frac{1}{2\pi} = N^2 \int_{-K/2}^{K/2} dk = N^2 K$

$\implies N = \frac{1}{\sqrt{2\pi K}}$, the same condition on N.

(d)



Define Δx as shown.

Then $\Delta x = 2 \times \frac{2\pi}{K} = \frac{4\pi}{K}$

Also $\Delta k = K$.

Thus $\Delta k\, \Delta x = 4\pi > 1$.

(2) $g(k) = \dfrac{N}{k^2+\alpha^2}$

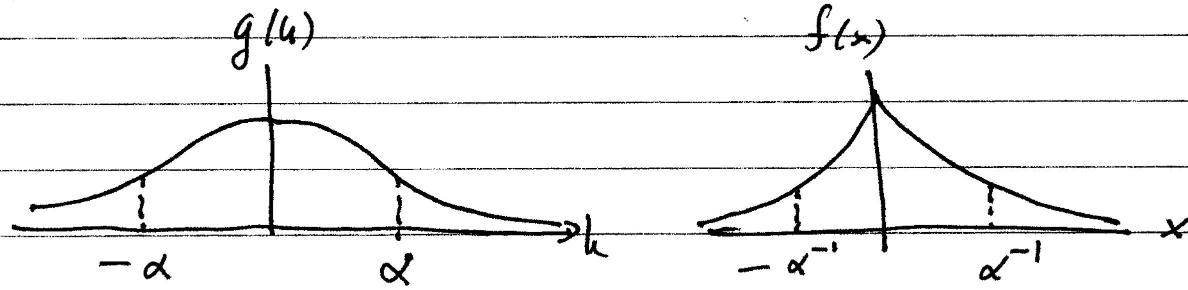
The form of $f(x)$ is $f(x) = \int_{-\infty}^{\infty} g(k) e^{ikx}\, dk$

$$= \int_{-\infty}^{\infty} \frac{N e^{ikx}}{k^2+\alpha^2}\, dk$$

Evaluate by contour integration. For $x>0$, close the contour above; the pole is at $k = i\alpha$, and

$$f(x) = 2\pi i \frac{N e^{i(i\alpha)x}}{2i\alpha} = \frac{N\pi}{\alpha} e^{-\alpha x}$$

$$\boxed{f(x) = \frac{N\pi}{\alpha} e^{-\alpha|x|}}$$



$\Delta k = 2\alpha$ and $\Delta x = 2\alpha^{-1}$,

so $\Delta k\, \Delta x = 4$, independent of α.

(5) Waves in a wave guide, $\lambda = \frac{c}{\sqrt{\nu^2 - \nu_0^2}}$.

Let $k = \frac{2\pi}{\lambda}$ and $\omega = 2\pi\nu$. The phase velocity is $v_{ph} = \frac{\omega}{k}$, and the group velocity is $v_{gr} = \frac{d\omega}{dk}$.

Group velocity

$$v_{gr} = \frac{d\omega}{dk} = \frac{2\pi\, d\nu}{-\frac{2\pi}{\lambda^2}\, d\lambda} = -\lambda^2 \frac{d\nu}{d\lambda} = \frac{-\lambda^2}{(d\lambda/d\nu)}$$

$$\frac{d\lambda}{d\nu} = \frac{-c\nu}{(\nu^2 - \nu_0^2)^{3/2}} = \frac{-c\nu}{(c/\lambda)^3} = \frac{-\nu\lambda^3}{c^2}$$

$$\therefore v_{gr} = \frac{\lambda^2 c^2}{\nu\lambda^3} = \frac{c^2}{\nu}\frac{\sqrt{\nu^2 - \nu_0^2}}{c}$$

$$\boxed{v_{gr} = c\sqrt{1 - \left(\frac{\nu_0}{\nu}\right)^2}}$$

Note that the group velocity is less than the speed of light.

GENERAL FORMULAS

Note: $\omega = 2\pi\nu$ and $k = 2\pi/\lambda$

$v_{phase} = \frac{\omega}{k} = \lambda\nu$

$v_{group} = \frac{d\omega}{dk} = -\lambda^2 \frac{d\nu}{d\lambda}$

⑥ Water waves.

- Surface tension waves have $\nu = \left(\frac{2\pi T}{\rho\lambda^3}\right)^{1/2}$.

$$v_{phase} = \lambda\nu = \left(\frac{2\pi T}{\rho\lambda}\right)^{1/2}$$

$$v_{group} = -\lambda^2 \frac{d\nu}{d\lambda} = -\lambda^2\left(\frac{2\pi T}{\rho}\right)^{1/2}\left(-\frac{3}{2}\lambda^{-5/2}\right) = \frac{3}{2}\left(\frac{2\pi T}{\rho\lambda}\right)^{1/2}$$

$$\boxed{v_{group} = \frac{3}{2} v_{phase}} \quad \text{(surface tension)}$$

- Gravity waves have $\nu = \left(\frac{g}{2\pi\lambda}\right)^{1/2}$.

$$v_{phase} = \lambda\nu = \left(\frac{g\lambda}{2\pi}\right)^{1/2}$$

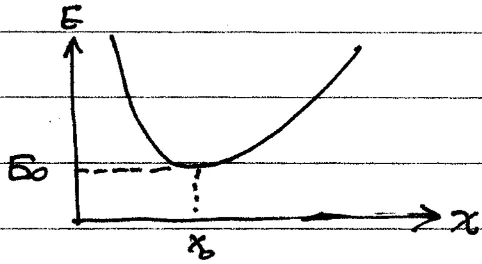
$$v_{group} = -\lambda^2 \frac{d\nu}{d\lambda} = -\lambda^2\left(\frac{g}{2\pi}\right)^{1/2}\left(-\frac{1}{2}\lambda^{-3/2}\right) = \frac{1}{2}\left(\frac{g\lambda}{2\pi}\right)^{1/2}$$

$$\boxed{v_{group} = \frac{1}{2} v_{phase}} \quad \text{(gravity)}$$

(7) Use the method on pages 36-38.

$E = p^2/2m + \frac{1}{2} m\omega^2 x^2$, and $p \sim \hbar/x$ by uncertainty relation.

Thus $E = \frac{\hbar^2}{2mx^2} + \frac{1}{2} m\omega^2 x^2$



The minimum energy is at $x = x_0$,

where $\frac{dE}{dx} = 0 = \frac{-\hbar^2}{mx^3} + m\omega^2 x$

$\therefore x_0 = \left(\frac{\hbar}{m\omega}\right)^{1/2}$

The ground state energy is estimated to be

$$E_0 = \frac{\hbar^2}{2m} \frac{m\omega}{\hbar} + \frac{1}{2} m\omega^2 \frac{\hbar}{m\omega} = \hbar\omega.$$

$\boxed{E_0 = \hbar\omega.}$ In fact, the ground state energy is $\frac{1}{2}\hbar\omega_0$.

(9) Use the method of pages 36 - 38.

Nuclear size $R = 10^{-12}$ cm $= 10$ fm.

Suppose the electron is initially confined inside the nucleus. Then by the uncertainty principle, its momentum must be

$$p \sim \frac{\hbar}{R} = \frac{\hbar c}{Rc} = \frac{197 \text{ MeV-fm}}{10 \text{ fm } \times c} = 20 \text{ MeV}/c.$$

But that implies the energy $E = \sqrt{m^2c^4 + p^2c^2}$ would be of order 20 MeV, much larger than the observed energy $E = 1$ MeV. Therefore the electron cannot have been confined inside the nucleus, initially.