Set 3 (due Wed, Sept. 19)
Problems from Chapter 3

(2) $\psi(x,t)$ obeys $i\hbar \frac{\partial \psi}{\partial t} = \frac{-\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V\psi$;

Therefore also $-i\hbar \frac{\partial \psi^*}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi^*}{\partial x^2} + V\psi^*$.

(V is real.) Now consider the probability current

$$j(x) = \frac{\hbar}{2mi} \left(\psi^* \frac{\partial \psi}{\partial x} - \frac{\partial \psi^*}{\partial x} \psi \right)$$

$$\frac{\partial j}{\partial x} = \frac{\hbar}{2mi} \left(\psi^* \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi^*}{\partial x^2} \psi \right)$$

$$= \frac{\hbar}{2mi} \left(\frac{-2m}{\hbar^2} \right) \left[\psi^* \left(i\hbar \frac{\partial \psi}{\partial t} - V\psi \right) - \left(-i\hbar \frac{\partial \psi^*}{\partial t} - V\psi^* \right) \psi \right]$$

$$= -\left[\psi^* \frac{\partial \psi}{\partial t} + \frac{\partial \psi^*}{\partial t} \psi \right] \qquad \text{(terms } \propto V \text{ cancel)}$$

$$= -\frac{\partial P}{\partial t} \quad \text{where} \quad P = \psi^* \psi .$$

This proves equation (3-12); ~~[illegible]~~ probability is conserved.

Problem 3

If V is complex, then, repeating the calculation in Problem 2,

$$\frac{\partial j}{\partial x} = -\left[\psi^*\left(\frac{\partial \psi}{\partial t} - \frac{V}{i\hbar}\psi\right) + \left(\frac{\partial \psi^*}{\partial t} + \frac{V^*}{i\hbar}\psi^*\right)\psi \right]$$

$$= -\frac{\partial P}{\partial t} + \frac{V - V^*}{i\hbar}\psi^*\psi$$

$$= -\frac{\partial P}{\partial t} + \frac{2}{\hbar}(\mathrm{Im}\, V)\, P \qquad \leftarrow (\text{Imaginary part} \equiv \mathrm{Im})$$

That is,

$$\frac{\partial P}{\partial t} = -\frac{\partial j}{\partial x} + \frac{2}{\hbar}(\mathrm{Im}\, V)\, P.$$

Also, integrating over x,

$$\frac{d}{dt}\int P(x,t)\, dx = \frac{2}{\hbar}\int (\mathrm{Im}\, V)\, P\, dx .$$

To describe <u>absorption</u> of particles, the imaginary part of V should be negative.

Problem 4 $\quad \psi(x) = \left(\frac{\pi}{\alpha}\right)^{-1/4} e^{-\alpha x^2/2}$

(a) $\langle x^n \rangle = \int_{-\infty}^{\infty} \psi^* x^n \psi \, dx = \left(\frac{\pi}{\alpha}\right)^{-1/2} \int_{-\infty}^{\infty} e^{-\alpha x^2} x^n \, dx.$

Note that $\langle x^n \rangle = 0$ if n is odd, because then the integrand is an odd function of x.

Integral identity for n even

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} x^n \, dx = \frac{(2k-1)!! \sqrt{\pi}}{2^k \sqrt{\alpha}\, \alpha^k} \quad \text{where } n = 2k.$$

Thus $\langle x^n \rangle = \frac{(2k-1)!!}{(2\alpha)^k}$ where $n = 2k$. $\boxed{= \frac{(n-1)!!}{(2\alpha)^{n/2}}}$

(b) $\langle x \rangle = 0$

$\langle x^2 \rangle = \frac{1}{2\alpha}$

Therefore, $\Delta x = \frac{1}{\sqrt{2\alpha}}.$

Problem 5

(a) $\phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int \psi(x)\, e^{-ipx/\hbar}\, dx$ (eq: 3-26)

$$= \frac{1}{\sqrt{2\pi\hbar}} \left(\frac{\pi}{\alpha}\right)^{-1/4} \int_{-\infty}^{\infty} e^{-\alpha x^2/2}\, e^{-ipx/\hbar}\, dx$$

Use this integral identity : $\int_{-\infty}^{\infty} e^{-\alpha x^2/2}\, e^{\beta x}\, dx = \sqrt{\frac{2\pi}{\alpha}}\, e^{\beta^2/2\alpha}$

Thus

$$\phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \left(\frac{\pi}{\alpha}\right)^{-1/4} \left(\frac{2\pi}{\alpha}\right)^{1/2} e^{-p^2/2\alpha\hbar^2} = \underbrace{(\pi\alpha\hbar^2)^{-1/4}\, e^{-p^2/2\alpha\hbar^2}}_{\text{p-space wave function.}}$$

(b) $\langle p^n \rangle = \int_{-\infty}^{\infty} |\phi(p)|^2\, p^n\, dp$

$= (\pi\alpha\hbar^2)^{-1/2} \int_{-\infty}^{\infty} e^{-p^2/\alpha\hbar^2}\, p^n\, dp$

If n is odd then the integral is 0, by symmetry.

(c) $\langle p \rangle = 0$ and $\langle p^2 \rangle = (\pi\alpha\hbar^2)^{-1/2} \int_{-\infty}^{\infty} e^{-p^2/\alpha\hbar^2}\, p^2\, dp$

$\langle p^2 \rangle = (\pi\alpha\hbar^2)^{-1/2}\, \frac{\sqrt{\pi}}{2}\, (\alpha\hbar^2)^{3/2}$ by the integral in Prob. 4

$\langle p^2 \rangle = \frac{1}{2}\alpha\hbar^2$

Thus $\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \sqrt{\frac{\alpha\hbar^2}{2}}$

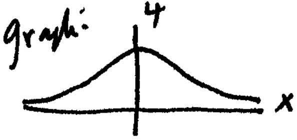
(d) The product of uncertainties is

$$\Delta p \cdot \Delta x = \sqrt{\frac{\alpha\hbar^2}{2}}\, \frac{1}{\sqrt{2\alpha}} = \frac{\hbar}{2}.$$

Problem 6

$$\psi(x) = \sqrt{\frac{2a^3}{\pi}} \frac{1}{x^2+a^2}$$

graph:



Expectation values of x and x^2

$\langle x \rangle = 0$ by symmetry $\left(\int_{-\infty}^{\infty} \frac{x\,dx}{(x^2+a^2)^2} = 0 \right)$

$$\langle x^2 \rangle = \frac{2a^3}{\pi} \int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+a^2)^2} = \frac{2a^3}{\pi} \frac{\pi}{2a} = a^2$$

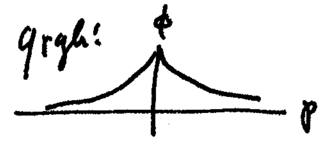
Momentum space wave function

$$\phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \psi(x)\, e^{-ipx/\hbar}\, dp$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \sqrt{\frac{2a^3}{\pi}} \frac{\pi}{a} e^{-|p|a/\hbar}$$ by contour integration

$$= \sqrt{\frac{a}{\hbar}}\, e^{-a|p|/\hbar}$$

graph:



Expectation values of p and p^2

$\langle p \rangle = 0$ by symmetry

$$\langle p^2 \rangle = \int |\phi(p)|^2 p^2 dp = \frac{a}{\hbar} \int_{-\infty}^{\infty} p^2 dp\, e^{-2a|p|/\hbar}$$

$$= \frac{2a}{\hbar} \int_0^{\infty} p^2 e^{-2ap/\hbar}\, dp$$

$$= \frac{2a}{\hbar} \frac{2}{(2a/\hbar)^3} = \frac{\hbar^2}{2a^2}$$

Identity:
$$\int_0^{\infty} u^2 e^{-\beta u} du = \frac{2}{\beta^3}$$

The product of uncertainties is

$$\Delta x \cdot \Delta p = a \cdot \sqrt{\frac{\hbar^2}{2a^2}} = \frac{\hbar}{\sqrt{2}}$$

Problem 7 Consider $\psi(x) = R(x)\, e^{iS(x)/\hbar}$

where $R(x)$ and $S(x)$ are real functions.

$$\underline{\langle p \rangle} = \int \psi^* \frac{\hbar}{i}\frac{\partial \psi}{\partial x}\, dx$$

$$= \int R e^{-iS/\hbar}\, \frac{\hbar}{i}\left[\frac{dR}{dx} + R\frac{i}{\hbar}\frac{dS}{dx}\right] e^{iS/\hbar}\, dx$$

$$= \int \frac{\hbar}{i} R \frac{dR}{dx}\, dx + \int R^2 \frac{dS}{dx}\, dx.$$

The first integral is 0, because

$$\int_{-\infty}^{\infty} R\frac{dR}{dx}\, dx = \frac{1}{2}R^2\Big|_{x=-\infty}^{x=+\infty} = 0.$$

(Note that $R \to 0$ as $x \to \infty$.) Thus $\boxed{\langle p \rangle = \int_{-\infty}^{\infty} R^2 \frac{dS}{dx}\, dx}$

$$\underline{\langle p^2 \rangle} = \int \psi^* (-\hbar^2)\frac{d^2\psi}{dx^2}\, dx$$

$$= -\hbar^2 \int R e^{-iS/\hbar}\left[\frac{d^2R}{dx^2} + \frac{i}{\hbar}\frac{dR}{dx}\frac{dS}{dx} + \frac{i}{\hbar}R\frac{d^2S}{dx^2} + \left(\frac{dR}{dx} + \frac{i}{\hbar}R\frac{dS}{dx}\right)\frac{i}{\hbar}\frac{dS}{dx}\right] e^{iS/\hbar}\, dx$$

$$= -\hbar^2 \int \left[R\frac{d^2R}{dx^2} + \underbrace{\frac{i}{\hbar}\left(2R\frac{dR}{dx}\frac{dS}{dx} + R^2\frac{d^2S}{dx^2}\right)} - \frac{1}{\hbar^2}R^2\left(\frac{dS}{dx}\right)^2\right] dx$$

This term integrates to 0,

because $= \frac{i}{\hbar}\frac{d}{dx}\left(R^2\frac{dS}{dx}\right)$ and $R^2\frac{dS}{dx} \to 0$ at ∞.

Also $\int R\frac{d^2R}{dx^2}\, dx = \int \left[\underbrace{\frac{d}{dx}\left(R\frac{dR}{dx}\right)} - \left(\frac{dR}{dx}\right)^2\right] dx$

integral is 0, similarly.

Thus

$$\boxed{\langle p^2 \rangle = \int_{-\infty}^{\infty}\left[\hbar^2\left(\frac{dR}{dx}\right)^2 + R^2\left(\frac{dS}{dx}\right)^2\right] dx}$$

Problem 4

```
Int[k_] :=
 Integrate[Exp[-a*x^2]*x^(2 k), {x, -Infinity, Infinity}, Assumptions → {a > 0}]
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Table[Int[k], {k, 0, 4}]
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$\left\{\frac{\sqrt{\pi}}{\sqrt{a}}, \frac{\sqrt{\pi}}{2\, a^{3/2}}, \frac{3\sqrt{\pi}}{4\, a^{5/2}}, \frac{15\sqrt{\pi}}{8\, a^{7/2}}, \frac{105\sqrt{\pi}}{16\, a^{9/2}}\right\}$

Note:

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} x^{2k}\, dx = \frac{(2k-1)!!\,\sqrt{\pi}}{2^k \sqrt{\alpha}\,\alpha^k}$$

Problem 5

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Integrate[Exp[-a*x^2/2] Exp[b*x], {x, -Infinity, Infinity},
 Assumptions → {a > 0}]
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$\frac{e^{\frac{b^2}{2a}}\sqrt{2\pi}}{\sqrt{a}}$

Note:

$$\int_{-\infty}^{\infty} e^{-ax^2/2} e^{bx}\, dx = \sqrt{\frac{2\pi}{a}}\, e^{b^2/2a}$$

Problem 6

```
Integrate[x^2*(x^2+a^2)^(-2), {x, -Infinity, Infinity},
 Assumptions → {a > 0}]
Integrate[u^2 Exp[-b*u], {u, 0, Infinity},
 Assumptions -> {b > 0}]
```

$\frac{\pi}{2\,a}$

$\frac{2}{b^3}$

Note:

$$\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+a^2)^2} = \frac{\pi}{2a}$$

$$\int_{0}^{\infty} u^2 e^{-bu}\, du = \frac{2}{b^3}$$