Set 4 due Sept. 26

① Linear operators : a, b, f
Non linear : c, d, e

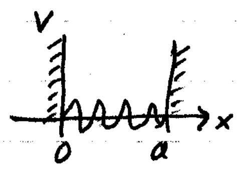
② The eigenvalue problem is $O_6 \psi = \lambda \psi$ where $\lambda =$ constant.
That is, $\int_{-\infty}^{x} \psi(x')\, x'\, dx' = \lambda \psi(x)$.
Differentiate both sides, $\psi(x)\, x = \lambda \frac{d\psi}{dx}$; and solve by separation of variables:

$$\frac{d\psi}{\psi} = \frac{x\,dx}{\lambda}$$

$$\ln \psi = \frac{x^2}{2\lambda} + c$$

$$\boxed{\psi(x) = N e^{x^2/2\lambda}}$$

The eigenfunction is square integrable if λ is negative, $\lambda < 0$.

④ 

$u_n(x) = \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a}$ where $0 \le x \le a$.
The uncertainty of x is $\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$

(The statement in the book is wrong.) The calculation of Δx is done in <u>problem 15</u>. The result is

$$(\Delta x)^2 = \frac{a^2}{12} - \frac{a^2}{2\pi^2 n^2}.$$

The uncertainty of p is $\Delta p = \sqrt{\langle p^2\rangle}$ because $\langle p\rangle = 0$.
$\langle p^2\rangle$ is given in equation (4-26). The result is

$$(\Delta p)^2 = \langle p^2\rangle = \frac{\hbar^2 \pi^2 n^2}{a^2}.$$

Thus

$$\Delta p \cdot \Delta x = \frac{\hbar \pi n}{a} \sqrt{\frac{a^2}{12} - \frac{a^2}{2\pi^2 n^2}} = \hbar \sqrt{\frac{\pi^2 n^2}{12} - \frac{1}{2}}.$$

Note that $\Delta p \cdot \Delta p \ge \hbar \sqrt{\frac{\pi^2}{12} - \frac{1}{2}}$ and it increases with n.

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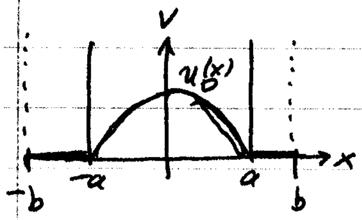
(5)



The initial wave function is

$$u_0(x) = \begin{cases} \sqrt{\frac{1}{a}} \cos \frac{\pi x}{2a} & \text{for } -a \le x \le a \\ 0 & \text{for } x < -a \text{ or } x > a. \end{cases}$$

NSW $(-a, a)$ increases suddenly to $(-b, b)$.

New ground state

$$u_0'(x) = \sqrt{\frac{1}{b}} \cos \frac{\pi x}{2b} \quad \text{for } -b \le x \le b.$$

The overlap of u_0' and u_0 is

$$A_0 = \int u_0^*(x)\, u_0'(x)\, dx = \sqrt{\frac{1}{a}}\sqrt{\frac{1}{b}} \int_{-a}^{a} \cos\frac{\pi x}{2a} \cos\frac{\pi x}{2b}\, dx$$

$$= \frac{1}{\sqrt{ab}} \int_{-a}^{a} \frac{1}{2}\left[\cos\frac{\pi x}{2}\left(\frac{1}{a}+\frac{1}{b}\right) + \cos\frac{\pi x}{2}\left(\frac{1}{a}-\frac{1}{b}\right)\right] dx$$

$$= \frac{1}{2\sqrt{ab}}\left[\frac{2ab}{\pi(a+b)} \sin\frac{\pi x}{2}\left(\frac{1}{a}+\frac{1}{b}\right)\Big|_{-a}^{a} + \frac{2ab}{\pi(b-a)} \sin\frac{\pi x}{2}\left(\frac{1}{a}-\frac{1}{b}\right)\Big|_{-a}^{a} \right]$$

$$= \frac{\sqrt{ab}}{\pi(a+b)} \sin\left(\frac{\pi}{2}+\frac{\pi a}{2b}\right)\times 2 + \frac{\sqrt{ab}}{\pi(b-a)} \sin\left(\frac{\pi}{2}-\frac{\pi a}{2b}\right)\times 2$$

$$A_0 = \frac{2\sqrt{ab}}{\pi}\,\frac{2b}{b^2-a^2}\cos\frac{\pi a}{2b} = \frac{4\,a^{1/2}b^{3/2}}{\pi\,(b^2-a^2)}\cos\frac{\pi a}{2b}$$

The probability that the particle will be found in the new ground state after the change is

$$P_0 = |A_0|^2 = \frac{16ab^3}{\pi^2(b^2-a^2)^2}\cos^2\frac{\pi a}{2b}$$

New excited state

$$u_1'(x) = \sqrt{\frac{1}{b}} \sin\frac{\pi x}{b} \quad \text{for } -b \le x \le b.$$

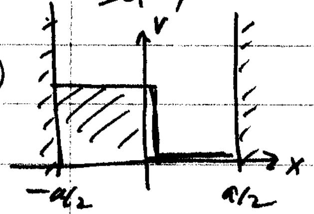
The overlap of u_1' and u_0 is

$$A_1 = \int u_0^*\, u_1'\, dx = \sqrt{\frac{1}{a}}\sqrt{\frac{1}{b}}\int_{-a}^{a} \cos\frac{\pi x}{2a}\sin\frac{\pi x}{b}\, dx$$

$= 0$ because the integrand is an odd function of x.

$P_1 = |A_1|^2 = 0$. The probability is 0 because u_0 and u_1' have opposite parities.

Set 4

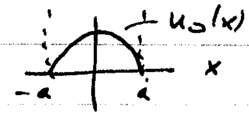


(6) The initial wave function is

$$\Psi(x,0) = \begin{cases} \sqrt{2/a} & \text{for } -\frac{a}{2} \le x \le 0 \\ 0 & \text{otherwise} \end{cases}$$

(a) The particle will not remain localized.

(b) <u>Ground state</u> $\quad u_0 = \sqrt{\frac{2}{a}} \cos \frac{\pi x}{a}$



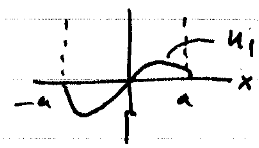
The overlap of the wavefunctions is

$$A_0 = \int u_0(x)\, \Psi(x,0)\, dx = \frac{2}{a}\int_{-a/2}^{0} \cos\frac{\pi x}{a}\, dx$$

$$= \frac{2}{a}\; \frac{a}{\pi} \sin\frac{\pi x}{a}\Big|_{-a/2}^{0} = \frac{2}{\pi}\left(0-(-1)\right) = \frac{2}{\pi}$$

The probability that an energy measurement would yield the ground state energy is $P_0 = |A_0|^2 = \frac{4}{\pi^2}$.

<u>First excited state</u> $\quad u_1 = \sqrt{\frac{2}{a}} \sin \frac{2\pi x}{a}$



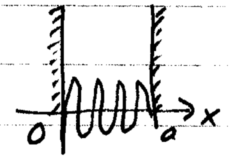
The overlap of the wave functions is

$$A_1 = \int u_1(x)\, \Psi(x,0)\, dx = \frac{2}{a}\int_{-a/2}^{0} \sin\frac{2\pi x}{a}\, dx$$

$$= \frac{2}{a}\left[\frac{-a}{2\pi}\cos\frac{2\pi x}{a}\right]_{-a/2}^{0} = \frac{-1}{\pi}\left[1-(-1)\right] = \frac{-2}{\pi}$$

The probability that an energy measurement would yield the 1st excited state energy is $P_1 = |A_1|^2 = \frac{4}{\pi^2}$.

15



$$u_n(x) = \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a}$$

$$\langle x \rangle = \frac{2}{a} \int_0^a \underbrace{\sin^2 \frac{n\pi x}{a}}_{\frac{1}{2}\left[1 - \cos \frac{2n\pi x}{a}\right]} \, x \, dx = \frac{1}{a} \int_0^a x\,dx - \frac{1}{a} \int_0^a \underbrace{\cos \frac{2n\pi x}{a}}_{\frac{a}{2\pi n} \frac{d}{dx} \sin\left(\frac{2\pi n x}{a}\right)} x\,dx$$

integrate by parts

$$\langle x \rangle = \frac{1}{2} a - \frac{1}{2\pi n} \left\{ \underbrace{\sin \frac{2\pi n x}{a} \, x \Big|_0^a}_{=0} - \underbrace{\int_0^a \sin \frac{2\pi n x}{a} \, dx}_{\text{integral is 0 by symmetry}} \right\}$$

$\langle x \rangle = \frac{a}{2}$ (obviously!)

$$\langle x^2 \rangle = \frac{2}{a} \int_0^a \sin^2 \frac{n\pi x}{a} \, x^2 \, dx = \frac{1}{a} \int_0^a x^2 dx - \frac{1}{a} \int_0^a \cos \frac{2\pi n x}{a} \, x^2 dx$$

(integrate by parts)

$$= \frac{a^2}{3} - \frac{1}{2\pi n} \left\{ \underbrace{\sin \frac{2\pi n x}{a} \, x^2 \Big|_0^a}_{0} - \int_0^a \underbrace{\sin \frac{2\pi n x}{a}}_{\frac{-a}{2\pi n} \frac{d}{dx} \cos \frac{2\pi n x}{a}} \, 2x \, dx \right\}$$

integrate by parts

$$= \frac{a^2}{3} - \frac{a}{(2\pi n)^2} \left\{ \cos \frac{2\pi n x}{a} \, 2x \Big|_0^a - \int_0^a \underbrace{\sin \frac{2\pi n x}{a} \, 2\,dx}_{\text{integral} = 0 \text{ by periodicity}} \right\}$$

$$= \frac{a^2}{3} - \frac{2a^2}{4\pi^2 n^2} = \frac{a^2}{3} - \frac{a^2}{2\pi^2 n^2}$$

The uncertainty of x is Δx where

$$(\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2 = \frac{a^2}{3} - \frac{a^2}{2\pi^2 n^2} - \frac{a^2}{4}$$

$$= \frac{a^2}{12} - \frac{a^2}{2\pi^2 n^2}$$

$$\boxed{\Delta x = a \sqrt{\frac{1}{12} - \frac{1}{2\pi^2 n^2}}}$$

Note that $\Delta x \to \frac{a}{\sqrt{12}}$ as $n \to \infty$, which is the uncertainty for constant probability density.