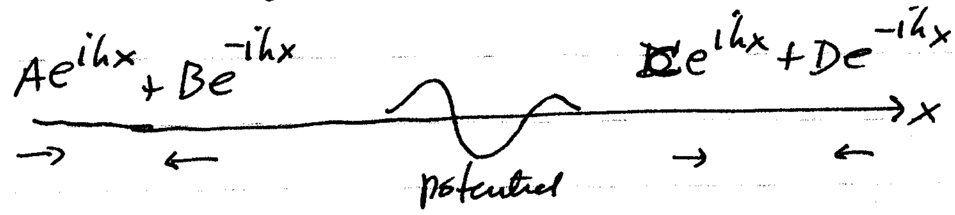
Phy 471, homework set 5

Chapter 5 of Gasiorowicz

① 

Define the scattering matrix S_{ij} by

$$\begin{pmatrix} C \\ B \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} A \\ D \end{pmatrix} \qquad (1)$$

↑ outgoing waves   ↑ incoming waves

The probability current in region 1 is

$$j_1 = \frac{\hbar}{2im}\left(\psi^* \frac{\partial \psi}{\partial x} - \frac{\partial \psi^*}{\partial x}\psi\right)$$

$$= \frac{\hbar}{2im}\Big[(A^* e^{-ikx} + B^* e^{ikx})\, ik\, (Ae^{ikx} - Be^{-ikx}) - (-ik)(A^* e^{-ikx} - B^* e^{ikx})(Ae^{ikx} + Be^{-ikx})\Big]$$

$$= \frac{\hbar k}{m}\left(|A|^2 - |B|^2\right);$$

and similarly, in region 2,

$$j_2 = \frac{\hbar k}{m}\left(|C|^2 - |D|^2\right).$$

These currents must be equal, by conservation of probability; therefore,

$$|A|^2 + |D|^2 = |C|^2 + |B|^2. \qquad (2)$$

Now calculate the right-hand side of (2) using (1).

$$|C|^2 + |B|^2 = |S_{11}A + S_{12}D|^2 + |S_{21}A + S_{22}D|^2$$

$$= \{ |S_{11}|^2 + |S_{21}|^2 \} |A|^2 + \{ |S_{12}|^2 + |S_{22}|^2 \} |D|^2$$
$$+ (S_{11}^* S_{12} + S_{21}^* S_{22}) A^* D$$
$$+ (S_{11} S_{12}^* + S_{21} S_{22}^*) A D^*$$

This must equal $|A|^2 + |D|^2$ (for any A and D) so the coefficients must be 1, 1, 0, 0; i.e.,

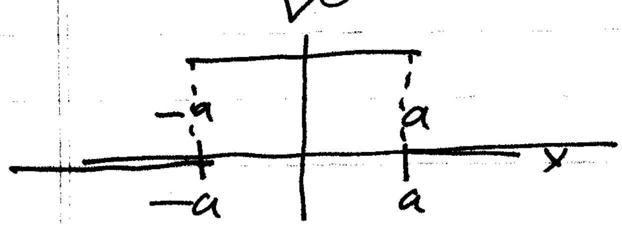
$$|S_{11}|^2 + |S_{21}|^2 = 1$$
$$|S_{12}|^2 + |S_{22}|^2 = 1$$
$$S_{11}^* S_{12} + S_{21}^* S_{22} = 0$$

The matrix S is unitary:

$$S^\dagger S = \begin{pmatrix} S_{11}^* & S_{21}^* \\ S_{12}^* & S_{22}^* \end{pmatrix} \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$$
$$= \begin{pmatrix} |S_{11}|^2 + |S_{21}|^2 & S_{11}^* S_{12} + S_{21}^* S_{22} \\ S_{12}^* S_{11} + S_{22}^* S_{21} & |S_{12}|^2 + |S_{22}|^2 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$S^\dagger S = \mathbb{1}$ (definition of unitarity)

② Scattering matrix of a potential well or barrier



$$V(x) = \begin{cases} V_0 & \text{for } -a<x<a \\ 0 & \text{for } x<-a \text{ or } x>a. \end{cases}$$

The wave function for energy E is

$$u(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & \text{for } x<-a \text{ w/ } k=\sqrt{2mE/\hbar^2} \\ Me^{iqx} + Ne^{-iqx} & \text{for } -a<x<a \text{ w/ } q=\sqrt{\frac{2m}{\hbar^2}(E-V_0)} \\ Ce^{ikx} + De^{-ikx} & \text{for } x>a \end{cases}$$

The continuity conditions — $u(x)$ and $u'(x)$ must be continuous at $x=-a$ and $x=a$.
There are 4 conditions:

$\varphi = ka$

$\theta = qa$

(1) $$A e^{-i\varphi} + Be^{i\varphi} = Me^{-i\theta} + Ne^{i\theta}$$

(2) $$k\left(Ae^{-i\varphi} - Be^{i\varphi}\right) = q\left(Me^{-i\theta} - Ne^{i\theta}\right)$$

(3) $$Me^{i\theta} + Ne^{-i\theta} = Ce^{i\varphi} + De^{-i\varphi}$$

(4) $$q\left(Me^{i\theta} - Ne^{-i\theta}\right) = k\left(Ce^{i\varphi} - De^{-i\varphi}\right)$$

There are 4 equations with 6 unknowns. Regard A and D as given. Solve for C and B.
Then determine the matrix from

$$C = S_{11} A + S_{12} D$$

$$B = S_{21} A + S_{22} D$$

① Find C as a function of A and D.
Eliminate B by adding $k \times (1) + (2)$

$$2kAe^{-i\varphi} = (k+q)Me^{-i\theta} + (k-q)Ne^{i\theta}$$

Substitute for M and N from equations (3) and (4):

$$2qMe^{i\theta} = (q+k)Ce^{i\varphi} + (q-k)De^{-i\varphi}$$
$$2qNe^{-i\theta} = (q-k)Ce^{i\varphi} + (q+k)De^{-i\varphi}$$

Thus

$$2kAe^{-i\varphi} = (k+q)e^{-i\theta}\frac{e^{-i\theta}}{2q}\left[(q+k)Ce^{i\varphi} + (q-k)De^{-i\varphi}\right]$$
$$+ (k-q)e^{i\theta}\frac{e^{i\theta}}{2q}\left[(q-k)Ce^{i\varphi} + (q+k)De^{-i\varphi}\right]$$

$$4kqAe^{-2i\varphi} = C\left[(k+q)^2e^{-2i\theta} - (k-q)^2e^{2i\theta}\right]$$
$$+ D\left[(q^2-k^2)e^{-2i\theta} + (k^2-q^2)e^{2i\theta}\right]e^{-2i\varphi}$$

Compare to $C = S_{11}A + S_{12}D$. Determines S_{11} and S_{12}:

$$S_{11} = \frac{4kq\,e^{-2i\varphi}}{(k+q)^2e^{-2i\theta} - (k-q)^2e^{2i\theta}}$$

$$= \frac{4kq\,e^{-2i\varphi}}{-(k^2+q^2)\,2i\sin 2\theta + 4kq\cos 2\theta}$$

$$\boxed{S_{11} = \frac{2kq\,e^{-2i\varphi}}{2kq\cos 2\theta - i(k^2+q^2)\sin 2\theta}}$$

and

$$S_{12} = \frac{-(k^2-q^2)\, 2i \sin 2\theta \;\; e^{-2i\varphi}}{(k+q)^2 e^{-2i\theta} - (k-q)^2 e^{2i\theta}}$$

$$\boxed{S_{12} = \frac{i(q^2-k^2)\sin 2\theta \;\; e^{-2i\varphi}}{2kq\cos 2\theta - i(k^2+q^2)\sin 2\theta}}$$

We are now half-way done.

(2) Find B as a function of A & D.
Eliminate C using equations (3) and (4) ($k\times(3)-(4)$)

$$2kDe^{-i\varphi} = (k-q)Me^{i\theta} + (k+q)Ne^{-i\theta}$$

This time substitute for M and N from eqs. (1) and (2)

$$2qMe^{-i\theta} = (q+k)Ae^{-i\varphi} + (q-k)Be^{i\varphi}$$
$$2qNe^{i\theta} = (q-k)Ae^{-i\varphi} + (q+k)Be^{i\varphi}$$

Thus

$$2kDe^{-i\varphi} = (k-q)\frac{e^{2i\theta}}{2q}\left\{(q+k)Ae^{-i\varphi} + (q-k)Be^{i\varphi}\right\} + (k+q)\frac{e^{-2i\theta}}{2q}\left\{(q-k)Ae^{-i\varphi} + (q+k)Be^{i\varphi}\right\}$$

$$4kqDe^{-2i\varphi} = B\left[-e^{2i\theta}(q-k)^2 + e^{-2i\theta}(q+k)^2\right] + A\left[e^{2i\theta}(k^2-q^2) + e^{-2i\theta}(q^2-k^2)\right]e^{-2i\varphi}$$

Compare to $B = S_{21}A + S_{22}D$.

Deduce S_{21} and S_{22}

$$S_{21} = \frac{(q^2-k^2)\, 2i \sin 2\theta \; e^{-2i\varphi}}{-e^{2i\theta}(q-k)^2 + e^{-2i\theta}(q+k)^2}$$

$$= \frac{i(q^2-k^2) \sin 2\theta \; e^{-2i\varphi}}{-(q^2+k^2)\, i \sin 2\theta + 2kq \cos 2\theta}$$

$$\boxed{S_{21} = \frac{i(q^2-k^2) \sin 2\theta \; e^{-2i\varphi}}{2kq \cos 2\theta - i(k^2+q^2) \sin 2\theta}}$$

Note that $S_{21} = S_{12}$.

$$S_{22} = \frac{4kq\, e^{-2i\varphi}}{-e^{2i\theta}(q-k)^2 + e^{-2i\theta}(q+k)^2}$$

$$\boxed{S_{22} = \frac{2kq\, e^{-2i\varphi}}{2kq \cos 2\theta - i(q^2+k^2) \sin 2\theta}}$$

Note that $S_{22} = S_{11}$.

Now verify the properties in Problem 1.

$$|S_{11}|^2 + |S_{21}|^2 = \frac{4k^2q^2 + (q^2-k^2)^2 \sin^2 2\theta}{4k^2q^2 \cos^2 2\theta + (k^2+q^2)^2 \sin^2 2\theta}$$

$$= \frac{4k^2q^2 + (q^2-k^2)^2 \sin^2 2\theta}{4k^2q^2 + (k^2-q^2)^2 \sin^2 2\theta} = 1. \checkmark$$

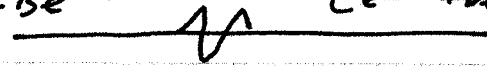
$|S_{22}|^2 + |S_{12}|^2 = 1$ in the same way. $\checkmark$

$$S_{11}^* S_{12} + S_{21}^* S_{22} = \frac{2kq\; i(q^2-k^2)\sin 2\theta + (-i)(q^2-k^2)\sin 2\theta\; 2kq}{[2kq \cos 2\theta + i(q^2+k^2)\sin 2\theta][2kq\cos 2\theta - i(k^2+q^2)\sin 2\theta]}$$

$$= 0. \checkmark$$

(3) Start with the definition of $S(k)$ from Exercise 1

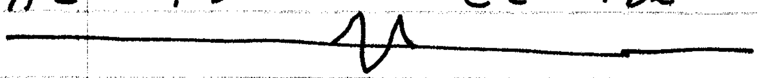
$Ae^{ikx} + Be^{-ikx}$ $\qquad$ $Ce^{ikx} + De^{-ikx}$



$$\begin{pmatrix} C \\ B \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} A \\ D \end{pmatrix}$$

Now change k to $-k$

$A'e^{-ikx} + B'e^{ikx}$ $\qquad$ $C'e^{-ikx} + D'e^{ikx}$



$$\begin{pmatrix} C' \\ B' \end{pmatrix} = \begin{pmatrix} S_{11}(-k) & S_{12}(-k) \\ S_{21}(-k) & S_{22}(-k) \end{pmatrix} \begin{pmatrix} A' \\ D' \end{pmatrix} \qquad \text{Eq 1}$$

But now note that case 2 is same as case 1 with the replacements $A \to B'$, $B \to A'$ and $C \to D'$, $D \to C'$

Hence

$$\begin{pmatrix} D' \\ A' \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} B' \\ C' \end{pmatrix} \quad \text{where } S_{ij} \text{ means } S_{ij}(k).$$

Now recall that $S^{+}S = 1$ so $S^{-1} = S^{+}$. Thus

$$\begin{pmatrix} B' \\ C' \end{pmatrix} = S^{+} \begin{pmatrix} D' \\ A' \end{pmatrix} = \begin{pmatrix} S_{11}^{*} & S_{21}^{*} \\ S_{12}^{*} & S_{22}^{*} \end{pmatrix} \begin{pmatrix} D' \\ A' \end{pmatrix} \qquad \text{Eq 2}$$

Compare Eq 1 and Eq 2:

$$C' = S_{11}(-k)A' + S_{12}(-k)D' = S_{12}^{*} D' + S_{22}^{*} A'$$

$$B' = S_{21}(-k)A' + S_{22}(-k)D' = S_{11}^{*} D' + S_{21}^{*} A'$$

Thus, equating the coefficients,

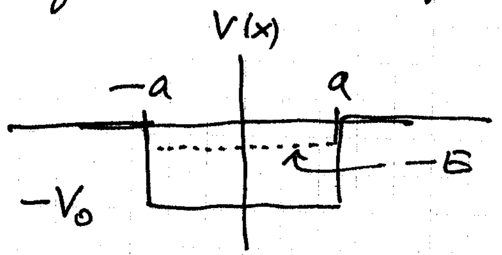
$$S_{11}(-k) = S_{22}^{*}(k) \qquad S_{12}(-k) = S_{12}^{*}(k)$$

$$S_{21}(-k) = S_{21}^{*}(k) \qquad S_{22}(-k) = S_{11}^{*}(k)$$

The author is wrong.

$$S(-k) = E\, S^{+}(k)\, E \quad \text{where } E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \text{exchange matrix.}$$

④ We will use the odd parity solution of a square well for the nuclear bound state.



Parameters

$a = 1.4 \times 10^{-13}$ cm

$E = -2.2$ MeV

$m = 0.8 \times 10^{-24}$ g (= reduced mass of p and n)

The problem is to find V_0 (= depth of the well).

Solution The eigenvalue equation is

$$\frac{\sqrt{\lambda - y^2}}{y} = -\cot y \quad \text{or} \quad \tan y = \frac{-y}{\sqrt{\lambda - y^2}}. \quad ①$$

Here $\lambda = \frac{2ma^2V_0}{\hbar^2}$, $y = qa$, and $E = -V_0 + \frac{\hbar^2 q^2}{2m}$ ②

The strategy will be to calculate $V_0 = \frac{\hbar^2 q^2}{2m} - E = \frac{\hbar^2 y^2}{2ma^2} - E$.

Note that $\lambda - y^2$ is known:

$$\lambda - y^2 = \frac{2ma^2}{\hbar^2}V_0 - \frac{2ma^2}{\hbar^2}(E + V_0) = \frac{2ma^2}{\hbar^2}(-E)$$

$$= \frac{2\,(0.8\times10^{-24}\,\text{g})\,(1.4\times10^{-13}\,\text{cm})^2}{(1.055\times10^{-27}\,\text{erg-s})^2} \times \frac{2.2\,\text{MeV} \times 1.6\times10^{-6}\,\text{erg}}{\text{MeV}}$$

$= 0.099.$

Now solve equation ① $\tan y = \frac{-y}{0.099}$.

The solution is $y = 1.631$. Plug that result into ②.

$$V_0 = \frac{\hbar^2 y^2}{2ma^2} - E = \frac{(1.055\times10^{-27}\,\text{erg-s})^2\,(1.631)^2}{2\,(0.8\times10^{-24}\,\text{g})(1.4\times10^{-13}\,\text{cm})^2}\,\frac{1\,\text{MeV}}{1.6\times10^{-6}\,\text{erg}} + 2.2\,\text{MeV}$$

$$\boxed{V_0 = 61.2 \text{ MeV.}}$$