Chapter 5

Problem 6 Eq. (5-26) is $T = \dfrac{e^{-2ika}\, 2kq}{2kq\cos 2qa - i(q^2+k^2)\sin 2qa}$

Eqs. (5-63) are $\begin{cases} \kappa = q \tan qa \\ \kappa = -q \cot qa \end{cases}$

In the denominator of T, let k be replaced by $i\kappa$ and set the denominator $=0$; that is,

$$0 = 2i\kappa q \underbrace{\cos 2qa}_{\cos^2 qa - \sin^2 qa} - i(q^2-\kappa^2)\underbrace{\sin 2qa}_{2\sin qa \cos qa}$$

Divide by $\cos^2 qa \Rightarrow$

$$0 = \kappa q(1-\tan^2 qa) - (q^2-\kappa^2)\tan qa$$

$$0 = \kappa q \tan^2 qa + (q^2-\kappa^2)\tan qa - \kappa q$$

Solve for $\tan qa$ by the quadratic ~~equation~~ formula

$$\tan qa = \frac{1}{2\kappa q}\left\{-(q^2-\kappa^2) \pm \sqrt{(q^2-\kappa^2)^2 + 4\kappa^2 q^2}\right\}$$

$$= \frac{1}{2\kappa q}\left\{-(q^2-\kappa^2) \pm (q^2+\kappa^2)\right\}$$

$$= \begin{cases} \kappa/q & \leftarrow \text{same as upper eq. 5-63} \\ -q/\kappa & \leftarrow \text{same as lower eq. 5-63} \end{cases}$$

So the denominator $=0$ for $k = i\kappa$ is same as condition for bound state energy. This is not an accident. Eq (5-26) is a solution for $u(x)$ of the form

$$u(x) = \begin{cases} e^{ikx} + Re^{-ikx} = e^{-\kappa x} + Re^{\kappa x} & \text{for } x < -a \\ Te^{ikx} = Te^{-\kappa x} & \text{for } x > a \end{cases}$$

This is a bound state provided R and $T \to \infty$ so that the non-normalizable term $e^{-\kappa x}$ can be neglected.

Problem 9 — Harmonic oscillator states

$$u_n(x) = C_n\, H_n(\alpha x)\, e^{-\frac{1}{2}\alpha^2 x^2} \quad \text{where} \quad \alpha = \sqrt{\frac{m\omega}{\hbar}}$$

$\boxed{n=0}$

$$u_0(x) = C_0\, e^{-\frac{1}{2}\alpha^2 x^2}$$

Normalization $\int u_0^2\, dx = C_0^2 \int_{-\infty}^{\infty} e^{-\alpha^2 x^2} dx = C_0^2 \frac{\sqrt{\pi}}{\alpha}$

$= 1$ so $C_0 = \left(\frac{\alpha^2}{\pi}\right)^{1/4}$

$\boxed{n=1}$

$$u_1(x) = C_1\, \alpha x\, e^{-\frac{1}{2}\alpha^2 x^2}$$

Normalization: $\int u_1^2\, dx = C_1^2 \alpha^2 \int_{-\infty}^{\infty} x^2 e^{-\alpha^2 x^2} dx = C_1^2 \frac{\sqrt{\pi}}{2\alpha}$

$= 1$ so $C_1 = \left(\frac{4\alpha^2}{\pi}\right)^{1/4}$

$\boxed{n=2}$

$$u_2(x) = C_2\, (2\alpha^2 x^2 - 1)\, e^{-\frac{1}{2}\alpha^2 x^2}$$

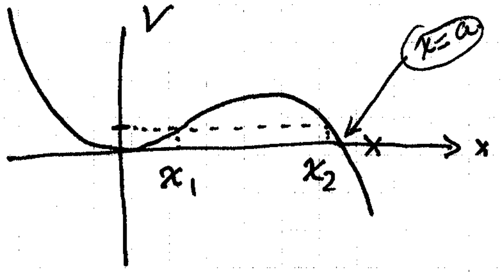
Normalization: $\int u_2^2\, dx = C_2^2 \int (4\alpha^4 x^4 - 4\alpha^2 x^2 + 1) e^{-\alpha^2 x^2}$

$= C_2^2 \left[4\alpha^4 \frac{3\sqrt{\pi}}{4\alpha^5} - 4\alpha^2 \frac{\sqrt{\pi}}{2\alpha^3} + \frac{\sqrt{\pi}}{\alpha} \right] = C_2^2 \frac{2\sqrt{\pi}}{\alpha}$

$= 1$ so $C_2 = \left(\frac{\alpha^2}{4\pi}\right)^{1/4}$.

Problem 10

$V(x) = \frac{1}{2} m\omega^2 \left(x^2 - \frac{x^3}{a}\right)$ where a is small.



Consider this as a tunneling problem. The particle hits the barrier with frequency $\frac{\omega}{2\pi}$.

The tunneling probability is $|T|^2$ given by Eq. (5-43)

$$|T|^2 = e^{-2\int_{x_1}^{x_2} \sqrt{(2m/\hbar^2)(V(x)-E)}\,dx}$$

The "turning points" x_1 and x_2 (= boundaries of the tunneling region) are solutions of $V(x) = E$.

Let $E = 0$ for purposes of estimation. (area under the curve is almost the same). Then

$$-\frac{1}{2}\ln|T|^2 = \int_0^a \sqrt{\frac{2m}{\hbar^2}\frac{m\omega^2}{2}\left(x^2 - \frac{x^3}{a}\right)}\,dx$$

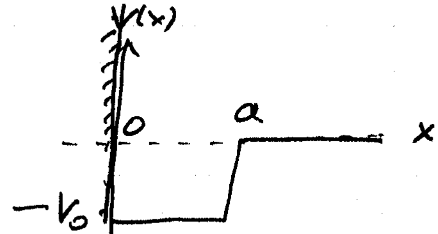
$$= \frac{m\omega}{\hbar} \underbrace{\int_0^a \sqrt{x^2 - x^3/a}\,dx}_{4a^2/15} = \frac{4m\omega a^2}{15\hbar}$$

The tunneling rate is $\text{Rate} = \frac{\omega}{2\pi} |T|^2$;

the mean lifetime is

$$\tau = \frac{1}{\text{Rate}} = \frac{2\pi}{\omega} e^{+\frac{8m\omega a^2}{15\hbar}}.$$

Problem 13



The wave function must be 0 at $x=0$, so in the well, $u(x) = A \sin qx$. $(x<a)$

where $E = -V_0 + \frac{\hbar^2 q^2}{2m}$. The outer region has incident and scattered waves, so $u(x) = e^{-ikx} + R e^{ikx}$ $(x<0)$

where $E = \frac{\hbar^2 k^2}{2m}$. The continuity conditions are

$$e^{-ika} + R e^{ika} = A \sin qa$$

$$-ik\left(e^{-ika} - R e^{ika}\right) = +qA \cos qa$$

Solve for R by dividing $2 \div 1$

$$\frac{-ik\left(e^{-ika} - R e^{ika}\right)}{e^{-ika} + R e^{ika}} = +q \cot qa$$

$$-ik\left(1 - R e^{2ika}\right) = q \cot qa \left(1 + R e^{2ika}\right)$$

$$R = \frac{ik + q \cot qa}{ik - q \cot qa}\, e^{-2ika}$$

Note that $\frac{ia+b}{ia-b} = \frac{\sqrt{a^2+b^2}\, e^{i\varphi}}{-\sqrt{a^2+b^2}\, e^{-i\varphi}}$ where $\varphi = \arctan \frac{a}{b}$

$= -e^{2i\varphi}$

Thus

$$R = -e^{2i\varphi} e^{-2ika} \quad \text{where } \varphi = \arctan\left(\frac{k}{q \cot qa}\right)$$

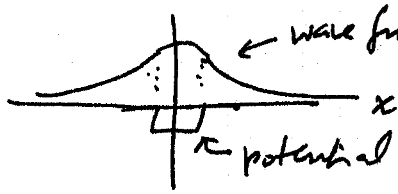
Note that $R = C e^{i\delta}$ where $\boxed{C = 1}$

and

$$\delta = 2\varphi - 2ka + \pi$$

$$\boxed{\delta = 2 \arctan\left(\frac{k}{q \cot qa}\right) - 2ka + \pi}$$

Problem 14



← Wave function of an electron with small binding energy

If V_0 is small then the binding energy is small, and the electron wave function extends to large x.

Therefore Δx is much larger than a.

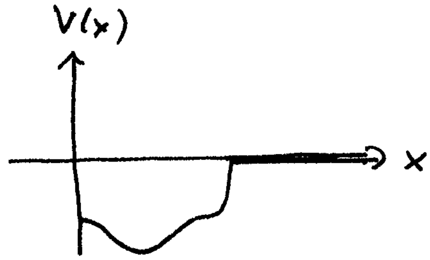
The kinetic energy is

$$\frac{p^2}{2m} \sim \frac{\hbar^2}{2m(\Delta x)^2}$$

and this is smaller than V_0 if Δx is large.

Then $E = \frac{p^2}{2m} - V_0$ is negative.

Problem 15



Given

$$f(E) \equiv \frac{1}{u(a)} \frac{du}{dx}\bigg|_{x=a}$$

(a) For a bound state, the wave function for $x > a$ is $u_2(x) = C e^{-\kappa x}$ where $E_B = -\frac{\hbar^2 \kappa^2}{2m}$. The continuity conditions are

$$u(a) = u_2(a) = C e^{-\kappa a}$$

$$u'(a) = u_2'(a) = -\kappa C e^{-\kappa a}$$

Therefore $\frac{u'(a)}{u(a)} = f(E) = -\kappa$

The bound state energy is

$$E_B = -\frac{\hbar^2 \kappa^2}{2m} = \boxed{-\frac{\hbar^2}{2m}\left[f(E_B)\right]^2}$$

(b) For a scattering state, the wave function for $x > a$ is $u_2(x) = e^{-ikx} + R e^{ikx}$ where $E = \frac{\hbar^2 k^2}{2m}$.

The continuity conditions are

$$u(a) = e^{-ika} + R e^{ika}$$

$$u'(a) = -ik\left(e^{-ika} - R e^{ika}\right)$$

thus

$$\frac{u'(a)}{u(a)} = f(E) = \frac{-ik\left(e^{-ika} - R e^{ika}\right)}{\left(e^{-ika} + R e^{ika}\right)}$$

Solve for R! The result is

$$\boxed{R = \frac{ik + f(E)}{ik - f(E)} e^{-2ika}}$$

Note that $|R|^2 = \frac{k^2 + |f|^2}{k^2 + |f|^2} = 1.$