PHY 471 homework set 7
Chapter 6

6-2 We will use these general properties of the Hermitian adjoint:

$$(\alpha A)^\dagger = \alpha^* A^\dagger \qquad (\alpha = \text{constant})$$
$$(A^\dagger)^\dagger = A$$
$$(AB)^\dagger = B^\dagger A^\dagger$$

① Consider $C = A + A^\dagger$.
Then $C^\dagger = A^\dagger + A = C$ so C is Hermitian.

② Consider $C = i(A - A^\dagger)$
Then $C^\dagger = (-i)(A^\dagger - A) = i(A - A^\dagger) = C$
so C is Hermitian.

③ Consider $C = AA^\dagger$.
Then $C^\dagger = (AA^\dagger)^\dagger = (A^\dagger)^\dagger A^\dagger = AA^\dagger = C$.
so C is Hermitian.

6-4

Consider $f(\lambda) \equiv \langle \psi + \lambda\phi | \psi + \lambda\phi \rangle$

$$= \langle\psi|\psi\rangle + \lambda^* \langle\phi|\psi\rangle + \lambda\langle\psi|\phi\rangle + |\lambda|^2\langle\phi|\phi\rangle.$$

Note that $f(\lambda) \geq 0$. Calculate the minimum of $f(\lambda)$; set $\frac{\partial f}{\partial \lambda} = 0$ where the partial derivative means differentiation w.r.t. λ with λ^* fixed:

$$\frac{\partial f}{\partial \lambda} = \langle\psi|\phi\rangle + \lambda^*\langle\phi|\phi\rangle$$

$$\therefore \lambda^* = -\frac{\langle\psi|\phi\rangle}{\langle\phi|\phi\rangle}.$$

Thus

$$f_{min} = \langle\psi|\psi\rangle - \frac{|\langle\phi|\psi\rangle|^2}{\langle\phi|\phi\rangle} - \frac{|\langle\phi|\psi\rangle|^2}{\langle\phi|\phi\rangle} + \frac{|\langle\phi|\psi\rangle|^2}{\langle\phi|\phi\rangle}$$

$$= \langle\psi|\psi\rangle - \frac{|\langle\phi|\psi\rangle|^2}{\langle\phi|\phi\rangle}$$

Since $f_{min} \geq 0$

$$\boxed{\langle\psi|\psi\rangle\langle\phi|\phi\rangle \geq |\langle\phi|\psi\rangle|^2}$$ Schwartz's inequality

6-5 $u_n(x) = \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a}$.

The completeness relation is $\sum_{n=1}^{\infty} u_n(x)\, u_n(y) = \delta(x-y)$.
Thus $\frac{2}{a} \sum_{n=1}^{\infty} \sin \frac{n\pi x}{a} \sin \frac{n\pi y}{a} = \delta(x-y)$.

6-8 Let U be unitary; i.e., $U^+U=1$.
Then

$$\langle U\psi | U\psi \rangle = \langle \psi | \underbrace{U^+U}_{=1} \psi \rangle = \langle \psi | \psi \rangle.$$

6-9 Let $U = e^{iA}$ where $A^+ = A$.
Write U as a power series

$$U = \sum_{n=0}^{\infty} \frac{i^n}{n!} A^n$$

$$U^+ = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} (A^+)^n = e^{-iA^+} = e^{-iA}$$

Then

$$U^+U = e^{-iA} e^{iA} = 1,$$

so U is unitary.

6-13 $\quad H = \dfrac{p^2}{2m} - eE_0 \cos\omega t \; x$

Position $\quad \dfrac{d\langle x\rangle}{dt} = \dfrac{i}{\hbar}\langle [H, x]\rangle \quad$ (see eq. 6-68)

Note $[H,x] = \left[\dfrac{p^2}{2m}, x\right] + [\underbrace{-eE_0 \cos\omega t \; x, x}_{=0}]$

$$= \frac{1}{2m}\left\{ p^2 x - pxp + pxp - xp^2 \right\}$$

$$= \frac{1}{2m}\left\{ p\underbrace{[p,x]}_{\hbar/i} + \underbrace{[p,x]}_{\hbar/i}\, p \right\} = \frac{\hbar}{i}\,\frac{p}{m}$$

Thus $\boxed{\dfrac{d\langle x\rangle}{dt} = \dfrac{\langle p\rangle}{m}} \quad$ (see eq. 6-72)

Momentum $\quad \dfrac{d\langle p\rangle}{dt} = \dfrac{i}{\hbar}\langle [H, p]\rangle$

Note $[H,p] = \underbrace{\left[\dfrac{p^2}{2m}, p\right]}_{=0} - eE_0 \cos\omega t \underbrace{[x,p]}_{i\hbar}$

Thus $\boxed{\dfrac{d\langle p\rangle}{dt} = eE_0 \cos\omega t}$

Energy $\quad \dfrac{d\langle H\rangle}{dt} = \left\langle \dfrac{\partial H}{\partial t}\right\rangle + \dfrac{i}{\hbar}\langle \underbrace{[H, H]}_{=0}\rangle$

$$\boxed{\frac{d\langle H\rangle}{dt} = eE_0\,\omega \sin\omega t \; \langle x\rangle}$$