Homework Solutions $\boxed{\text{Set 9}}$

Chapter 7 problems, due Wed. Oct 31

7-3 Recall $x = \sqrt{\frac{\hbar}{2m\omega}}\,(A + A^{\dagger})$

where $A|n\rangle = \sqrt{n}\,|n-1\rangle$

and $A^{\dagger}|n\rangle = \sqrt{n+1}\,|n+1\rangle$

The eigenstates $|n\rangle$ are orthonormal.

The matrix elements of x

$$\langle n|x|m\rangle = \sqrt{\frac{\hbar}{2m\omega}}\,\langle n|A + A^{\dagger}|m\rangle$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\,\Big\{ \underbrace{\langle n|\sqrt{m}\,|m-1\rangle}_{\sqrt{m}\,\delta(n,\,m-1)} + \underbrace{\langle n|\sqrt{m+1}\,|m+1\rangle}_{\sqrt{m+1}\,\delta(n,\,m+1)} \Big\}$$

(by orthonormality)

$$= \sqrt{\frac{\hbar}{2m\omega}}\,\Big\{ \sqrt{m}\,\delta(n,\,m-1) + \sqrt{m+1}\,\delta(n,\,m+1) \Big\}$$

Note that $\langle n|x|m\rangle = 0$ unless $n = m \pm 1$.

7-4 Recall $p = -i\sqrt{\frac{m\omega\hbar}{2}}\,(A - A^{\dagger})$

By the same method as problem 7-3, the matrix elements of p are

$$\langle n|p|m\rangle = -i\sqrt{\frac{m\omega\hbar}{2}}\,\Big\{ \sqrt{m}\,\delta(n,\,m-1) - \sqrt{m+1}\,\delta(n,\,m+1) \Big\}.$$

by completeness

7-5

$$\langle m | px | n \rangle = \sum_{k=0}^{\infty} \langle m | p | k \rangle \langle k | x | n \rangle$$

$$= \sum_{k=0}^{\infty} (-i) \sqrt{\frac{m\omega\hbar}{2}} \sqrt{\frac{\hbar}{2m\omega}}$$

using result of 7-4

$$\left\{ \sqrt{k}\, \delta(m, k-1) - \sqrt{k+1}\, \delta(m, k+1) \right\}$$

using result of 7-3

$$\left\{ \sqrt{n}\, \delta(k, n-1) + \sqrt{n+1}\, \delta(k, n+1) \right\}$$

Identity : $\sum_k \delta(k,a)\, \delta(k,b) = \delta(a,b)$

0 unless $k=a$ so $\sum_k$ reduces to 1 term.

$\therefore \langle m | px | n \rangle$

$$= \frac{\hbar}{2i} \left\{ \sqrt{m+1} \sqrt{n}\, \delta(m+1, n-1) + \sqrt{m+1} \sqrt{n+1}\, \delta(m+1, n+1) - \sqrt{m} \sqrt{n}\, \delta(m-1, n-1) - \sqrt{m} \sqrt{n+1}\, \delta(m-1, n+1) \right\}$$

7-6 $\langle m | xp | n \rangle = \sum_{k=0}^{\infty} (-i) \sqrt{\frac{m\omega\hbar}{2}} \sqrt{\frac{\hbar}{2m\omega}}$

$$\left\{ \sqrt{k}\, \delta(m, k-1) + \sqrt{k+1}\, \delta(m, k+1) \right\} \left\{ \sqrt{n}\, \delta(k, n-1) - \sqrt{n+1}\, \delta(k, n+1) \right\}$$

$$= \frac{\hbar}{2i} \left\{ \sqrt{m+1} \sqrt{n}\, \delta(m+1, n-1) - \sqrt{m+1} \sqrt{n+1}\, \delta(m+1, n+1) + \sqrt{m} \sqrt{n}\, \delta(m-1, n-1) - \sqrt{m} \sqrt{n+1}\, \delta(m-1, n+1) \right\}$$

7-7 Now compute $\langle m | [p, x] | n \rangle$

$= \langle m | px | n \rangle - \langle m | xp | n \rangle$ ← (These are known from problems 7-5 and 7-6)

$$= \frac{\hbar}{2i} \Big\{ 0 + 2\sqrt{m+1}\sqrt{n+1}\,\delta(m+1, n+1) - 2\sqrt{m}\sqrt{n}\,\delta(m-1, n-1) + 0 \Big\}$$

Note that $\delta(m+1, n+1) = \delta(m-1, n-1) = \delta(m, n)$.
Thus

$$\langle m | [p, x] | n \rangle = \frac{\hbar}{2i} \Big\{ 2(n+1) - 2n \Big\} \delta(m, n)$$

$$= \frac{\hbar}{i} \delta(m, n) \quad \underline{\text{as claimed}}.$$

7-13 Equation (7-63) is the equation of motion of an operator,

$$\frac{dB}{dt} = \frac{i}{\hbar} [H, B]. \qquad (7\text{-}63)$$

Consider $H = \frac{p^2}{2m} + mgx$. Then

$$\frac{dx}{dt} = \frac{i}{\hbar}[H, x] = \frac{i}{\hbar}\Big\{ \frac{1}{2m}[p^2, x] + mg \underbrace{[x, x]}_{0} \Big\}$$

$$[p^2, x] = ppx - pxp + pxp - xpp$$

$$= p \underbrace{[p, x]}_{\hbar/i} + \underbrace{[p, x]}_{\hbar/i} p = \frac{2\hbar}{i} p$$

$$\frac{dx}{dt} = \frac{i}{\hbar}\frac{1}{2m}\frac{2\hbar}{i}p = \frac{p}{m} \qquad \boxed{\frac{dx}{dt} = \frac{p}{m} \quad (1)}$$

and

$$\frac{dp}{dt} = \frac{i}{\hbar}\left\{\frac{1}{2m}\underbrace{[p^2, p]}_{=0} + mg\underbrace{[x, p]}_{=i\hbar}\right\}$$

$$= \frac{i}{\hbar}mg\,i\hbar = -mg \qquad \boxed{\frac{dp}{dt} = -mg \quad (2)}$$

The solution of (2) is

$$\boxed{p(t) = p(0) - mgt.}$$

Then (1) becomes $\frac{dx}{dt} = \frac{p(0)}{m} - gt$ which

has the solution $\boxed{x(t) = x(0) + \frac{p(0)}{m}t - \frac{1}{2}gt^2}$.

7-14

The Hamiltonian is $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 - e\mathcal{E}x$.

The first equation of motion is

$$\frac{dx}{dt} = \frac{i}{\hbar}[H, x] = \frac{p}{m}. \qquad (1)$$

The second equation of motion is

$$\frac{dp}{dt} = \frac{i}{\hbar}[H, p] = \frac{i}{\hbar}\left\{\frac{1}{2}m\omega^2[x^2, p] - e\mathcal{E}[x, p]\right\}$$

Here $[x, p] = i\hbar$

$$[x^2, p] = \underbrace{xxp - xpx}_{x[x,p]} + \underbrace{xpx - pxx}_{[x,p]x} = 2i\hbar x$$

$$\therefore \frac{dp}{dt} = \frac{i}{\hbar}\left\{ \frac{m\omega^2}{2}\, 2i\hbar x - e\mathcal{E} i\hbar \right\}$$

$$\frac{dp}{dt} = -m\omega^2 x + e\mathcal{E} \qquad (2)$$

Equations (1) and (2) are same as classical eqs.

<u>Solution</u>. Note that

$$\frac{d^2x}{dt^2} = \frac{1}{m}\,\overset{\text{by (1)}}{\frac{dp}{dt}} = \overset{\text{by (2)}}{-\omega^2 x + \frac{e\mathcal{E}}{m}}$$

The solution of this inhomogeneous linear equation is

$$x(t) = A\cos\omega t + B\sin\omega t + \frac{e\mathcal{E}}{m\omega^2}$$

(the third term is the "particular solution") where A and B are constant operators determined from the initial conditions. Also,

$$p(t) = m\frac{dx}{dt} = -m\omega A\sin\omega t + m\omega B\cos\omega t$$

<u>Initial values</u>

$$x(0) = A + \frac{e\mathcal{E}}{m\omega^2} \quad\Rightarrow\quad A = x(0) - \frac{e\mathcal{E}}{m\omega^2}$$

$$p(0) = m\omega B \quad\Rightarrow\quad B = \frac{p(0)}{m\omega}$$

So $\boxed{x(t) = x(0)\cos\omega t + \frac{p(0)}{m\omega}\sin\omega t + \frac{e\mathcal{E}}{m\omega^2}(1-\cos\omega t)}$

<u>Commutator at unequal times</u>

$$[x(t_1), x(t_2)] = \left[x(0)\cos\omega t_1 + \frac{p(0)}{m\omega}\sin\omega t_1,\ x(0)\cos\omega t_2 + \frac{p(0)}{m\omega}\sin\omega t_2\right]$$

$$= \frac{i\hbar}{m\omega}\cos\omega t_1\sin\omega t_2 - \frac{i\hbar}{m\omega}\sin\omega t_1\cos\omega t_2$$

$$= \frac{i\hbar}{m\omega}\sin[\omega(t_2-t_1)].$$