PHY-851: QUANTUM MECHANICS I
Final Exam /Total: 40 points/
December 11, 2001

NAME...

A. MULTIPLE CHOICE (*encircle the correct answers*) /12/.

1. Thermal neutrons (energy 0.025 eV) have their de Broglie wavelength equal to

   a. 2×10^{-5} cm; *b.* 2×10^{-8} cm; *c.* 2×10^{-13} cm.

2. For two possible states of the same particle, $\psi_1(x) = \exp(ikx - \alpha x^2)$ and $\psi_2(x) = \exp(ikx - 2\alpha x^2)$ where k and α are real constants, what is the ratio $\langle p_x \rangle_1 / \langle p_x \rangle_2$ of the expectation values of the momentum:

   $$a.\ \ \sqrt{2};\quad b.\ \ 1/\sqrt{2};\quad c.\ \ 2;\quad d.\ \ 1/2;\quad e.\ \ 1.$$

3. For a given barrier, the reflection coefficients at given energy are R_l for the incident wave coming from the left and R_r for the incident wave coming from the right, respectively. They satisfy the relation

   a. $R_l = R_r$; *b.* $R_l = -R_r$; *c.* $R_l + R_r = 1$; *d.* $R_l^2 + R_r^2 = 1$.

4. For the solution of the time-dependent Schrödinger equation to be fully determined, one needs to know the initial values of the

   a. wave function;

   b. its time derivative;

   c. both.

PROBLEM B. /15/

A particle is placed in a potential well of finite depth U_0. The width a of the well is such that the particle has only one bound state with binding energy $\epsilon = U_0/2$. Calculate the probability to find the particle in the classically forbidden region.

PROBLEM C. /13/

A particle of mass m is in a ground state of a one-dimensional potential $U(x) = (1/2)kx^2$. After a sudden capture of another particle, the mass changed $m \to m'$. What is the probability to find the new compound particle in an excited oscillator state?

SOLUTIONS

A. MULTIPLE CHOICE

1. *b*. Just because of the fact that the wavelength for thermal neutrons is of the order of a typical lattice period in solid crystals, one can use such neutrons for diffraction experiments in studies of crystal structure.
2. *e*. For a wave function in the form of $\psi(x) = e^{ikx} f(x)$ with *real* k and *real* square integrable $f(x)$, the expectation value of the momentum is $\hbar k$, independently of the form of $f(x)$. /To calculate the expectation value, one has to normalize $\psi(x)$./
3. *a*. See *Problem 4b, Homework 5.*
4. *a*.

PROBLEM B. In the square well,

$$U(x) = \begin{cases} 0, & |x| > a/2, \\ -U_0, & |x| < a/2, \end{cases} \tag{1}$$

the (unique) bound state is an even nodeless function of x,

$$\psi(x) = B\cos(kx), \quad k = \sqrt{\frac{2m(U_0 - \epsilon)}{\hbar^2}}, \quad |x| < a/2. \tag{2}$$

Outside the well, $|x| > a/2$, the wave function decays exponentially,

$$\psi(x) = Ae^{\pm\kappa x}, \quad \kappa = \sqrt{\frac{2m\epsilon}{\hbar^2}}. \tag{3}$$

In our conditions a great simplification comes from the fact that $\epsilon = U_0/2$,

$$k = \kappa = \sqrt{\frac{mU_0}{\hbar^2}}. \tag{4}$$

Due to certain parity, the matching can be done at $x = +a/2$ only,

$$B\cos(ka/2) = Ae^{-ka/2}, \quad -Bk\sin(ka/2) = -kAe^{-ka/2}. \tag{5}$$

The ratio of these equations gives the relation between the parameters of the problem

$$\tan(ka/2) = 1 \quad \rightsquigarrow \quad ka = \frac{\pi}{2}. \tag{6}$$

The probability to be outside the well is

$$P_{\text{out}} = 2\int_{a/2}^{\infty} dx\, A^2 e^{-2kx} = \frac{A^2}{k} e^{-ka}. \tag{7}$$

Because of eqs. (5) and (6), this is equal to

$$P_{\text{out}} = \frac{B^2}{k}\cos^2(ka/2) = \frac{B^2}{2k}. \tag{8}$$

The probability to be inside the well is

$$P_{\text{in}} = 2\int_0^{a/2} dx\, B^2\cos^2(kx) = \frac{B^2}{2k}[ka + \sin(ka)] = \frac{B^2}{2k}\left(\frac{\pi}{2} + 1\right). \tag{9}$$

Thus, we have the relation

$$\frac{P_{\text{out}}}{P_{\text{in}}} = \frac{2}{2+\pi}; \tag{10}$$

from $P_{\text{out}} + P_{\text{in}} = 1$ we come to

$$P_{\text{out}} = \frac{2}{4+\pi} = 0.28. \tag{11}$$

PROBLEM C. With the sudden perturbation, the wave function ψ_0 of the ground state of the Hamiltonian H does not have time to change but it is not a stationary state of the new Hamiltonian H',

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}k\hat{x}^2, \quad \hat{H}' = \frac{\hat{p}^2}{2m'} + \frac{1}{2}k\hat{x}^2. \tag{12}$$

The wave function is now a linear combination

$$\psi_0 = \sum_n C_n\psi'_n, \quad \sum_n |C_n|^2 = 1, \tag{13}$$

of the eigenstates ψ'_n of H', with the coefficients

$$C_n = \langle\psi'_n|\psi_0\rangle. \tag{14}$$

The probability to find the particle in any of the excited states ($n \neq 0$) is

$$P = \sum_{n\neq 0}|C_n|^2 = 1 - |C_0|^2. \tag{15}$$

Therefore it is sufficient to find the probability

$$P_0 = |C_0|^2 \tag{16}$$

of staying in the ground state after the perturbation. The overlap of the ground states can be found, for example, with the use of the coordinate representation. Taking into account that

$$m\omega = m\sqrt{\frac{k}{m}} = \sqrt{km}, \quad m'\omega' = \sqrt{km'}, \tag{17}$$

we find the corresponding ground state wave functions

$$\psi_0(x) = \frac{(km)^{1/8}}{(\pi\hbar)^{1/4}} e^{-(\sqrt{km}/2\hbar)x^2}, \tag{18}$$

$$\psi_0'(x) = \frac{(km')^{1/8}}{(\pi\hbar)^{1/4}} e^{-(\sqrt{km'}/2\hbar)x^2}, \tag{19}$$

and their overlap

$$C_0 = \int dx\, \psi_0'(x)\psi_0(x) = \frac{(mm')^{1/8}}{\sqrt{(\sqrt{m}+\sqrt{m'})/2}}, \tag{20}$$

$$P_0 = \frac{(mm')^{1/4}}{(\sqrt{m}+\sqrt{m'})/2}, \tag{21}$$

$$P = 1 - \frac{(mm')^{1/4}}{(\sqrt{m}+\sqrt{m'})/2}. \tag{22}$$

It is easy to show that always $P > 0$, as it should be.