PRACTICE PROBLEMS

1. An electron beam forms a diffraction pattern in the two slit experiment. Does each single electron create the same pattern?

2. A particle beam prepared with the initial coordinate spread $\Delta x(0)$ undergoes further quantum spreading during time t. How does its final spread $\Delta x(t)$ depend on the dispersion law (expression of energy $\epsilon(p)$ as a function of momentum)?

3. A charged quantum particle is moving in the uniform electric field. The expectation value of its radius vector is $\langle \mathbf{r}(t) \rangle$. Does this function coincide with the classical trajectory $\mathbf{r}(t)$ in the same field?

4. Let E_n be the energy of the n-th level. What is the n-dependence of E_n for $n \gg 1$ in

   a. harmonic oscillator field;

   b. square well with infinitely high walls;

   c. potential $U(x) \propto x^4$?

5. According to Ritz's empirical *combination rule*, all inverse wavelengths of radiation $1/\lambda$ observed in the spectrum of the hydrogen atom can be presented as proportional to the differences of two "spectral terms" $1/n^2$ and $1/n'^2$ with integer n and n',

$$\frac{1}{\lambda} = R_H \left(\frac{1}{n'^2} - \frac{1}{n^2} \right), \tag{1}$$

   and the *Rydberg constant* $R_H = 109678$ cm^{-1}. Explain this result by quantum mechanics.

6. An excited hydrogen atom sequentially emits two photons with the wavelengths 6563 Å and 1216 Å. Determine the main quantum numbers and binding energies of the initial, intermediate and final states.

7. A hydrogen atom, initially in the $2p$ state, emits a photon. Calculate the velocity of the recoil atom.

8. In classical electrodynamics, an electron bound in the hydrogen atom has to radiate electromagnetic waves with radiation frequencies being multiples of the electron revolution frequency. Show that the predictions of quantum and classical mechanics agree for the electron transitions between the levels with large quantum numbers $n', n \gg 1$ if $\Delta n = |n - n'| \ll n, n'$ ("*correspondence principle*"). /For simplicity consider circular classical orbits/.

9. Calculate the shift of the ground state energy for the electron in the atoms of heavy hydrogen isotopes deuterium and tritium compared to the normal hydrogen atom.

10. Calculate the radiation wavelength for the electron transition from the first excited state to the ground state in hydrogen, deuterium and positronium atoms (positronium is the hydrogen-like e^-e^+ state).

11. For the hydrogen-like ions He^+ and Li^{++} calculate
    a. the radius of the lowest Bohr orbit;
    b. the ionization potential for the ground state;
    c. the first excitation potential and the wavelength of the corresponding radiation.

12. A particle of mass m is moving in a one-dimensional potential $U(x)$ which is negative for all x and $U(x) \to 0$ at $x \to \pm\infty$. Does it always support at least one bound state? In the case it does, give a rough estimate of the spatial localization length of this state in terms of the characteristic depth U_0 and width a of the potential.

13. For an arbitrary linear operator $\hat{F}$ determine if the following operators are Hermitian:

$$\mathrm{a})\hat{F}^\dagger\hat{F}; \quad \mathrm{b})\hat{F}\hat{F}^\dagger; \quad \mathrm{c})\hat{F}-\hat{F}^\dagger; \quad \mathrm{d})\hat{F}+\hat{F}^\dagger.$$

    When the operator $\hat{F}^2$ is Hermitian?

14. The coordinate kernel $F(x,x')$ of an operator $\hat{F}$ have the form

$$\mathrm{a})f(x+x'); \quad \mathrm{b})f(x-x'); \quad \mathrm{c})f(x)g(x').$$

    What are the constraints for these functions if $\hat{F}$ is Hermitian?

15. Operators $\hat{A}$ and $\hat{B}$ are constants of motion. Is their commutator a constant of motion?

16. Calculate the commutation relations $[\hat{l}_j, \hat{r}_k]$, $[\hat{l}_j, \hat{p}_k]$ and $[\hat{l}_j, \hat{l}_k]$ where $\hat{\mathbf{l}} = [\hat{\mathbf{r}} \times \hat{\mathbf{p}}]/\hbar$ is the angular momentum operator in units of $\hbar$ and j, k are Cartesian coordinates (x, y, z) of vector operators.

17. Does the operator $\hat{L}_z^2$, z-component of the orbital momentum squared, commute with
    a) $\hat{p}_x$; *b)* $\hat{p}_x^2$; *c)* $\hat{p}_x^2 + \hat{p}_y^2$; *d)* $\hat{p}_x^2 - \hat{p}_y^2$,
    where $\hat{\mathbf{p}}$ is the momentum of a particle?

18. Find the uncertainty relation for the operators of coordinate $\hat{x}$ and an arbitrary smooth function of momentum $f(\hat{p})$.

19. An operator $\hat{A}$ satisfies $\hat{A}^4 = 1$. Find its eigenvalues if it is Hermitian and if it is not restricted to being Hermitian.

20. Two operators $\hat{A}$ and $\hat{B}$ anticommute, $\hat{A}\hat{B}+\hat{B}\hat{A} = 0$. Can they simultaneously have certain values? Give an example of anticommuting operators.

21. Write down the eigenfunction of the position vector $\hat{\mathbf{r}}$ corresponding to the eigenvalue $\mathbf{r}_0$ in the coordinate and momentum representations.

22. Write down the eigenfunction of the momentum operator $\hat{\mathbf{p}}$ corresponding to the eigenvalue $\mathbf{p}_0$ in the coordinate and momentum representations.

23. Can a unitary operator $\hat{U}$ be at the same time a projection operator?

24. The coordinate wave function of a particle is

$$\psi(x) = Ce^{(i/\hbar)qx}u(x),$$

where C is a constant, and $u(x)$ is a real square integrable function. Find the expectation value of the momentum in this state.

25. Two pieces of metal are placed close to each other so that the tails of the electron wave functions overlap. Assuming that these wave functions are

$$\psi_1(x) = A_1(x)\exp(i\phi_1) \quad \text{and} \quad \psi_2(x) = A_2(x)\exp(i\phi_2),$$

where $A_{1,2}(x)$ are real functions, ϕ_1 and ϕ_2 are constant but different phases, find the probability current through the interface.

26. Consider the Schrödinger equation where the potential is given by a complex function with the real part V and imaginary part W, the so-called optical potential $U(\mathbf{r}) = V(\mathbf{r}) + iW(\mathbf{r})$. Derive the continuity equation and interpret the results in relation to the sign of W.

27. For two particles of mass m_1 and m_2 introduce, instead of their position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$, their relative distance $\mathbf{r}$ and the center-of-mass coordinate $\mathbf{R}$. Find the corresponding momenta vectors $\mathbf{p}$ and $\mathbf{P}$, check the commutation relations of the components of $\mathbf{r}$ and $\mathbf{R}$ with $\mathbf{p}$ and $\mathbf{P}$.

28. A wave function of a particle of mass m in an infinitely deep potential box of size a is $\psi(x) = A\sin^2(\pi x/a)$. Find the distribution of probabilities of various values of particle energy, mean energy and the variance.

29. Find an approximate value of the ground state energy of the harmonic oscillator using the variational method with the trial function $\psi(x) = C(1+x^2/a^2)^{-s}$ where $s = 1$ or $s = 2$. Compare the results with the exact value. What value of s would give the best result?

30. Consider a particle of mass m in a potential bounded on the left by the infinitely high wall at $x = 0$, equal to $U(x) = -U_0, U_0 > 0$, for $x < a$, and $U(x) = 0$ for $x > a$. Find the interval of values U_0 where the well supports two bound states.

31. An electron is trapped in the potential well of depth U_0 and size $R = 10$ Å where it has only one bound state with the binding energy equal to $U_0/2$. Calculate the value of U_0, the probability density $|\psi|^2$ for the electron, find the coordinate of the maximum probability and draw qualitatively the function $\psi(x)$,

32. Estimate the probability for the proton and the neutron in the deuteron to be outside the region $r < R$ of nuclear attraction.

33. For the potential that consists of the infinitely high wall at $x = 0$ and the narrow well $-g\delta(x-a)$ with positive constants g and a, use the variational method with trial functions $\psi(x) = Ax\exp(-\beta x^s)$, $s = 1$ and $s = 2$, to find the conditions for the parameters (m, g, a), for which the bound state in this potential does exist. Compare with the exact solution.

34. Solve the Schrödinger equation for a slow particle reflected from the repulsive barrier (the potential is infinite for $x < 0$, equal to $U_0 > 0$ for $0 < x < a$ and zero outside this barrier); consider the cases when energy $0 < E < U_0$ or $E > 0$. Find the phase shift of the reflected wave. Consider the limiting transition to the infinitely high U_0. Consider the limiting cases of low and high energies (long and short wavelength limits with respect to the barrier size a).

35. Find the transmission coefficient for the rectangular potential, $U(x) = 0$ for $x < a$ and $x > b > a$, $U(x) = U_0$ for $a < x < b$. Consider $U_0 < 0$ and $U_0 > 0$. Discuss particular cases

$$\text{a) } E \gg U_0 > 0; \quad \text{b) } E \to 0.$$

Consider the limiting transition to the δ-potential.

36. Which of the following quantities: energy E, components of the momentum $\mathbf{p}$, components of the orbital momentum $\mathbf{l}$, its square $\mathbf{l}^2$, parity $\mathcal{P}$, are conserved when a particle is moving
a. with no external fields (free motion);
b. in the static uniform field along the z direction;
c. in the static central field $U(r)$;
d. in the field $U = f(\rho)$ where ρ is the radius in the xy-plane;
e. in the uniform field along the x-direction with the time-dependent amplitude?

37. A particle in the infinitely deep potential box of width a has an initial wave function $\Psi(x, t = 0) = A \sin^3(\pi x/a)$. Find the wave function at arbitrary time $t > 0$. Does the particle return to the initial state at some time T?

38. Find the Green function $G(\mathbf{p}, t; \mathbf{p}', t')$ in the momentum representation for a free particle.

39. A nucleus of atomic number A consists of Z protons and $N = A - Z$ neutrons. Assuming velocity-independent nucleon-nucleon forces, establish the energy weighted dipole sum rule.

40. For a system of N identical particles of mass m define the density operator

$$\hat{\rho}(\mathbf{r}) = \sum_{a=1}^{N} \delta(\mathbf{r} - \hat{\mathbf{r}}_a).$$

Find the density fluctuation operator $\hat{\rho}_{\mathbf{k}}$ as the Fourier-component of $\hat{\rho}(\mathbf{r})$ for the wave vector $\mathbf{k}$. Assuming velocity-independent forces between the particles, establish the energy weighted dipole sum rule for the operator $\hat{\rho}_{\mathbf{k}}$.

41. Find the uncertainty relation between the Heisenberg position operators at $t = 0$, $\hat{x}(0)$, and at arbitrary $t > 0$, $\hat{x}(t)$, for a harmonic oscillator.

42. The spatial inversion operator $\mathcal{P}$ transforms $x \to -x$ and $p \to -p$. Find the expression of $\mathcal{P}$ in terms of creation and annihilation operators.

43. For a coherent state $|\alpha\rangle$ of the harmonic oscillator, find
a. the coordinate wave function;
b. uncertainty of the number of quanta;
c. product of uncertainties $(\Delta x)(\Delta p)$.

44. Using the properties of creation and annihilation operators, find the eigenfunctions of the harmonic oscillator $\psi_n(x)$ with $n = 1$ and $n = 2$ in the coordinate representation.

45. Calculate matrix elements $\langle n'|x^s|n\rangle$ where $s = 3$ and $s = 4$; n and n' label the stationary states of the harmonic oscillator.

46. A particle with mass m and electric charge e is moving in the harmonic oscillator potential of frequency ω and static uniform electric field $\mathcal{E}$ applied along the axis x. Find
a. the energy spectrum E_n of stationary states $|n; \mathcal{E}\rangle$;
b. the induced dipole moment d_n;
c. the polarizability $(\partial d_n/\partial \mathcal{E})$;
d. the operator which transforms the states $|n; 0\rangle$ with no field into the

states $|n;\mathcal{E}\rangle$ in the presence of the field.
e. Show that the ground state $|0;\mathcal{E}\rangle$ is a coherent state, find the corresponding parameter α and the average number of quanta (defined with respect to the case with no field) in this state.

47. A particle of mass m is placed in the ground state in the field of the elastic force $-kx$. At the initial moment the restoring force is suddenly changes, $k \to k'$. What is the probability for the particle to remain in the ground state of the new potential?

48. For a particle in the uniform magnetic field $B = B_z$ show that the operators
$$\hat{\xi} = \hat{x} + \frac{\hat{v}_y}{\omega_c}, \quad \text{and} \quad \hat{\eta} = \hat{y} - \frac{\hat{v}_x}{\omega_c}$$
are constants of motion. What is the physical meaning of these quantities? Can they simultaneously have certain values? Find the spectrum of eigenvalues of the operator $\hat{\rho}^2 = \hat{\xi}^2 + \hat{\eta}^2$. What operator is the analog of the cyclotron radius of the orbit?

49. A particle of mass m is moving in the crossed uniform time-independent fields, magnetic $B = B_z$ and electric $\mathcal{E} = \mathcal{E}_x$. Find the energy spectrum of the stationary states.

50. A particle is moving in the uniform magnetic field $B = B_z$ and one-dimensional oscillator field $U(x) = (1/2)m\omega^2 x^2$. Find the energy spectrum of the stationary states. Are the levels degenerate?

51. Find an acceleration operator for a particle in an arbitrary static electromagnetic field $\mathbf{E}(\mathbf{r}), \mathbf{B}(\mathbf{r})$.