PHY-851 QUANTUM MECHANICS I
Homework 10, 50 points
November 14 - 28, 2001
Harmonic oscillator. Operator formalism.

Reading: *Merzbacher*, Chapter 5.

1. /7/ *Merzbacher*, Problem 3, p. 90.

2. /13/ *Merzbacher*, Problem 6, p. 91.

3. /6/ A particle of mass m and electric charge e is placed in the one-dimensional harmonic oscillator potential of frequency ω and the uniform electric field $\mathcal{E}$ along the same axis.

   a. Find the wave functions and the energy spectrum of the particle.

   b. With a particle in the ground state of the problem, at time $t = 0$ the electric field is suddenly switched off. Find the probability of finding the particle at $t > 0$ in the n^{th} stationary state of the oscillator.

   c. The ground state in the presence of the electric field acquires the nonzero expectation value of the electric dipole moment $\langle \hat{d} \rangle$ proportional to the applied field $\mathcal{E}$. Find the coefficient of proportionality (*static polarizability*).

4. /15/ Any linear operator $\hat{F}$ in the coordinate representation can be defined as an integral operator acting on an arbitrary function $\psi(x)$ according to

$$\hat{F}\psi(x) = \int dx' \, F(x, x')\psi(x'), \tag{1}$$

   where the function $F(x, x')$ is called the *kernel* of the operator.

   a. Construct kernels $F(x, x')$ corresponding to the operators $\hat{x}$, $\hat{p}$, inversion $\hat{\mathcal{P}}$, displacement $\hat{\mathcal{D}}(a)$ and scale transformation $\hat{\mathcal{M}}(\alpha)$; the last three operators are defined in *Problem 3, Homework 4.*

   b. Find the most general form of the kernel $F(x, x')$ for an operator $\hat{F}$ which commutes with the coordinate operator $\hat{x}$.

   c. Find the most general form of the kernel $F(x, x')$ for an operator $\hat{F}$ which commutes with the momentum operator $\hat{p}$.

   d. Find the most general form of the kernel $F(x, x')$ for an operator $\hat{F}$ which commutes with $\hat{x}$ *and* $\hat{p}$.

   e. Consider an operator $\hat{F}$ with the *factorized* kernel, $F(x, x') = f(x)g(x')$. At what condition the operator $\hat{F}$ is Hermitian? For a Hermitian operator of this type find its eigenfunctions and eigenvalues. Find the *degeneracies* of the eigenvalues (their multiplicities in the spectrum).

f. A particle is moving in the external, *nonlocal* but Hermitian, potential field $\hat{U}$ which is defined by the kernel $U(\mathbf{r}, \mathbf{r}')$. Is the continuity equation valid in this case? Is the normalization of the wave function conserved in time?

5. /9/ *a.* Let all eigenvalues of a Hermitian operator $\hat{F}$ in N-dimensional Hilbert space be different. Show that the N^{th} power, $\hat{F}^N$, of this operator can be expressed as a linear combination of the lower powers

$$\hat{F}^{N-1},\ \hat{F}^{N-2},\ \dots,\ \hat{F},\ \hat{1}.$$

As an example of the general result consider the inversion operator $\hat{\mathcal{P}}$.

b. Let $\{f_i\}$, $i = 1, \dots, N$, be the set of the eigenvalues of the same operator $\hat{F}$. Construct the projection operator $\hat{\Lambda}_i$ which projects out the state $|i\rangle$ with the value f_i of $\hat{F}$.

c. Show that the operator $\hat{F}$ with the factorized kernel, *Problem 3, e*, can be transformed into a projection operator $\hat{\Lambda} = c\hat{F}$ with multiplication by a certain constant c. Which state is projected out by the operator $\hat{\Lambda}$?