PROBLEM. A state of a particle of mass $m$ is described by the wave function
\[ \psi(r) = f(r)e^{i(k \cdot r)}, \]
where $f(r)$ is a real square integrable function, and $k$ is a real vector. Determine the probability density in the coordinate space and in the momentum space, probability current $j(r)$ and the expectation value of the momentum $\langle p \rangle$. 
SOLUTION. The probability density in coordinate space

\[ \rho(r) = N^2 |\psi(r)|^2 = N^2 f^2(r), \]

where \( N \) is the normalization factor,

\[ N^2 = \frac{1}{\int d^3 r f^2(r)}. \]

The normalization is possible because of the square integrability of the packet amplitude \( f(r) \). The momentum representation wave function

\[ \phi(p) = N \int d^3 r e^{-\frac{i}{\hbar} (p \cdot r)} \psi(r) = N \int d^3 r e^{-\frac{i}{\hbar} (p - \hbar k \cdot r)} f(r) = N f_{p - \hbar k}, \]

the Fourier-component of the wave packet amplitude \( f(r) \) corresponding to the momentum \( p - \hbar k \). The probability density in the momentum space

\[ \rho_p = |\phi(p)|^2 = N^2 |f_{p - \hbar k}|^2. \]

The coordinate normalization guarantees that the momentum wave function is normalized as well,

\[ \int \frac{d^3 p}{(2\pi \hbar)^3} |\phi(p)|^2 = N^2 \int d^3 p |f_{p - \hbar k}|^2 = N^2 \int d^3 p |f_p|^2 = 1. \]

In the last integral we can simply shift the integration variable \( p \to p + \hbar k \) without changing the infinite limits. The probability current can be expressed in terms of the gradient of the coordinate phase \( S(r)/\hbar = (k \cdot r) \),

\[ j(r) = \frac{\rho(r)}{m} \nabla S = \rho(r) \frac{\hbar k}{m}. \]

The expectation value of momentum

\[ \langle p \rangle = N^2 \int \frac{d^3 p}{(2\pi \hbar)^3} |f_{p - \hbar k}|^2 p = N^2 \int \frac{d^3 p}{(2\pi \hbar)^3} |f_p|^2 (p + \hbar k). \]

For a real function \( f(r) \) the Fourier-components satisfy

\[ f_{-p} = f_{p}^*, \]

Therefore \( |f_p|^2 = |f_{-p}|^2 \), and

\[ \int d^3 p |f_p|^2 p = 0, \]

a real packet does not create any direction of motion. Then eq. (8) gives, taking into account the normalization,

\[ \langle p \rangle = \hbar k. \]
The same result can be derived with the coordinate wave function:

\[ \langle p \rangle = N^2 \int d^3 r f(r)e^{-i(k \cdot r)}(-i\hbar \nabla)(e^{i(k \cdot r)} f(r)) = \]

\[ N^2 \int d^3 r f(r)e^{-i(k \cdot r)}[f\hbar k - (i\hbar \nabla)f(r)]e^{i(k \cdot r)}. \quad (12) \]

Due to the square integrability of \( f \), the integral

\[ \int d^3 r f \nabla f = \frac{1}{2} \int d^3 r \nabla(f^2) = \oint dA \cdot \nabla(f^2) \quad (13) \]

vanishes, and the result coincides with (10),

\[ \langle p \rangle = \hbar k N^2 \int d^3 r f^2(r) = \hbar k, \quad (14) \]

where we again used the normalization of the wave packet. The exact shape of the packet does not matter here although its quantum spreading depends on the shape.