PHY-851: QUANTUM MECHANICS I
Quiz 2
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PROBLEM. A state of a particle of mass m is described by the wave function

$$\psi(\mathbf{r}) = f(\mathbf{r})e^{i(\mathbf{k}\cdot\mathbf{r})}, \tag{1}$$

where $f(\mathbf{r})$ is a real square integrable function, and $\mathbf{k}$ is a real vector. Determine the probability density in the coordinate space and in the momentum space, probability current $\mathbf{j}(\mathbf{r})$ and the expectation value of the momentum $\langle\mathbf{p}\rangle$.

SOLUTION. The probability density in coordinate space

$$\rho(\mathbf{r}) = N^2|\psi(\mathbf{r})|^2 = N^2 f^2(\mathbf{r}), \tag{2}$$

where N is the normalization factor,

$$N^2 = \frac{1}{\int d^3r\, f^2(\mathbf{r})}. \tag{3}$$

The normalization is possible because of the square integrability of the packet amplitude $f(\mathbf{r})$. The momentum representation wave function

$$\phi(\mathbf{p}) = N \int d^3r\, e^{-(i/\hbar)(\mathbf{p}\cdot\mathbf{r})}\psi(\mathbf{r}) = N \int d^3r\, e^{-(i/\hbar)(\mathbf{p}-\hbar\mathbf{k})\cdot\mathbf{r}} f(\mathbf{r}) = N f_{\mathbf{p}-\hbar\mathbf{k}}, \tag{4}$$

the Fourier-component of the wave packet amplitude $f(\mathbf{r})$ corresponding to the momentum $\mathbf{p} - \hbar\mathbf{k}$. The probability density in the momentum space

$$\rho_{\mathbf{p}} = |\phi(\mathbf{p})|^2 = N^2|f_{\mathbf{p}-\hbar\mathbf{k}}|^2. \tag{5}$$

The coordinate normalization guarantees that the momentum wave function is normalized as well,

$$\int \frac{d^3p}{(2\pi\hbar)^3}|\phi(\mathbf{p})|^2 = N^2 \int d^3p\, |f_{\mathbf{p}-\hbar\mathbf{k}}|^2 = N^2 \int d^3p\, |f_{\mathbf{p}}|^2 = 1. \tag{6}$$

In the last integral we can simply shift the integration variable $\mathbf{p} \rightarrow \mathbf{p} + \hbar\mathbf{k}$ without changing the infinite limits. The probability current can be expressed in terms of the gradient of the coordinate phase $S(\mathbf{r})/\hbar = (\mathbf{k}\cdot\mathbf{r})$,

$$\mathbf{j}(\mathbf{r}) = \frac{\rho(\mathbf{r})}{m}\nabla S = \rho(\mathbf{r})\frac{\hbar\mathbf{k}}{m}. \tag{7}$$

The expectation value of momentum

$$\langle\mathbf{p}\rangle = N^2 \int \frac{d^3p}{(2\pi\hbar)^3}\, |f_{\mathbf{p}-\hbar\mathbf{k}}|^2\mathbf{p} = N^2 \int \frac{d^3p}{(2\pi\hbar)^3}\, |f_{\mathbf{p}}|^2(\mathbf{p} + \hbar\mathbf{k}). \tag{8}$$

For a real function $f(\mathbf{r})$ the Fourier-components satisfy

$$f_{-\mathbf{p}} = f^*_{\mathbf{p}}. \tag{9}$$

Therefore $|f_{\mathbf{p}}|^2 = |f_{-\mathbf{p}}|^2$, and

$$\int d^3p\, |f_{\mathbf{p}}|^2\mathbf{p} = 0, \tag{10}$$

a real packet does not create any direction of motion. Then eq. (8) gives, taking into account the normalization,

$$\langle\mathbf{p}\rangle = \hbar\mathbf{k}. \tag{11}$$

The same result can be derived with the coordinate wave function:

$$\langle \mathbf{p} \rangle = N^2 \int d^3r\, f(\mathbf{r}) e^{-i(\mathbf{k}\cdot\mathbf{r})} (-i\hbar\nabla)(e^{i(\mathbf{k}\cdot\mathbf{r})} f(\mathbf{r})) =$$

$$N^2 \int d^3r\, f(\mathbf{r}) e^{-i(\mathbf{k}\cdot\mathbf{r})} [f\hbar\mathbf{k} - (i\hbar\nabla) f(\mathbf{r})] e^{i(\mathbf{k}\cdot\mathbf{r})}. \tag{12}$$

Due to the square integrability of f, the integral

$$\int d^3r\, f\nabla f = \frac{1}{2}\int d^3r\, \nabla(f^2) = \oint d\mathcal{A}\cdot\nabla(f^2) \tag{13}$$

vanishes, and the result coincides with (10),

$$\langle \mathbf{p} \rangle = \hbar\mathbf{k}\, N^2 \int d^3r\, f^2(\mathbf{r}) = \hbar\mathbf{k}, \tag{14}$$

where we again used the normalization of the wave packet. The exact shape of the packet does not matter here although its quantum spreading depends on the shape.