PHY-851: QUANTUM MECHANICS I
Quiz 3
October 12, 2001

NAME....................................................

PROBLEM. Find all bound states for the particle of mass $m$ in the potential $U(x) = -g\delta(x)$, $g > 0$. 
SOLUTION. On both sides of the $\delta$-peak, the bound state wave function (negative energy $E = -\epsilon$) should be exponentially decaying,

$$\psi(x) = \begin{cases} Ae^{\kappa x}, & x < 0, \\ Be^{-\kappa x}, & x > 0, \end{cases}$$

(1)

where

$$\kappa = \sqrt{\frac{2m\epsilon}{\hbar^2}}.$$  

(2)

Since the potential is an even function of $x$, the solution should possess certain parity, $B = \pm A$, where the signs refer to positive and negative parity, $\Pi = \pm 1$. The matching condition for the wave function at the origin reads $\psi(0) = B = A$, - only positive parity is allowed. The derivative of the wave function has a discontinuity at the origin,

$$\psi'(0^+) - \psi'(0^-) = \frac{2m(-g)}{\hbar^2} \psi(0) \sim \kappa = \frac{mg}{\hbar^2}.$$  

(3)

We have found a unique value for energy: one and only one bound state does exist for any value of $g$,

$$E = -\epsilon = -\frac{h^2\kappa^2}{2m} = -\frac{mg^2}{2\hbar^2}.$$  

(4)

Since the $\delta$-potential can be considered as a limit of a potential well of depth $U_0$ and width $a$, when

$$U_0 \to \infty, \ a \to 0, \ U_0 a \to g = \text{const},$$

(5)

the solution (4) coincides with that found for a shallow well with binding energy

$$\epsilon = \frac{mU_0^2a^2}{2\hbar^2}.$$  

(6)