PHY-851: QUANTUM MECHANICS I
Quiz 3
October 12, 2001

NAME.....

PROBLEM. Find all bound states for the particle of mass m in the potential $U(x) = -g\delta(x),\ g > 0$.

SOLUTION. On both sides of the δ-peak, the bound state wave function (negative energy $E = -\epsilon$) should be exponentially decaying,

$$\psi(x) = \begin{cases} Ae^{\kappa x}, & x < 0, \\ Be^{-\kappa x}, & x > 0, \end{cases} \tag{1}$$

where

$$\kappa = \sqrt{\frac{2m\epsilon}{\hbar^2}}. \tag{2}$$

Since the potential is an even function of x, the solution should possess certain parity, $B = \pm A$, where the signs refer to positive and negative parity, $\Pi = \pm 1$. The matching condition for the wave function at the origin reads $\psi(0) = B = A$, - only positive parity is allowed. The derivative of the wave function has a discontinuity at the origin,

$$\psi'(0+) - \psi'(0-) = \frac{2m(-g)}{\hbar^2}\psi(0) \quad \rightsquigarrow \quad \kappa = \frac{mg}{\hbar^2}. \tag{3}$$

We have found a unique value for energy: one and only one bound state does exist for any value of g,

$$E = -\epsilon = -\frac{\hbar^2\kappa^2}{2m} = -\frac{mg^2}{2\hbar^2}. \tag{4}$$

Since the δ-potential can be considered as a limit of a potential well of depth U_0 and width a, when

$$U_0 \to \infty, \quad a \to 0, \quad U_0 a \to g = \text{const}, \tag{5}$$

the solution (4) coincides with that found for a shallow well with binding energy

$$\epsilon = \frac{mU_0^2 a^2}{2\hbar^2}. \tag{6}$$