SOLUTIONS for Homework #1

1. The photon energy for a given wave length λ equals

$$E_\gamma = \hbar\omega = \frac{2\pi\hbar c}{\lambda} \approx 60\,\text{keV}, \tag{1}$$

where it is useful to memorize a simple numerical relation

$$\hbar c = 197 \text{ MeV} \cdot \text{fm} \approx 2 \times 10^{-5} \text{ eV} \cdot \text{cm}. \tag{2}$$

Standard conservation laws for the Compton effect determine the energy transferred from the photon to the electron originally at rest,

$$\Delta E = \frac{E_\gamma}{1 + 2(E_\gamma/mc^2)\sin^2(\vartheta/2)} \approx 51\,\text{keV}, \tag{3}$$

where m is the electron mass, $mc^2 = 511$ keV, and ϑ is the photon scattering angle. The binding (ionization) energy of the lowest, the so-called K-, shell in the Mo atom, $Z = 42$, can be roughly estimated as

$$E_b = 1\,\text{Ry} \times Z^2 \approx 24\,\text{keV}. \tag{4}$$

Then eq. (3) determines the value of final kinetic energy of the electron,

$$E_{kin} = \Delta E - E_b \approx 27\,\text{keV}. \tag{5}$$

/A precise result is 31 keV since the empirical value of the wave length λ_K for the K-edge of the absorption of X-rays for Mo is $\lambda_K = 0.619\,\AA$ that corresponds to the ionization energy of 20 keV, rather than 24 keV of our estimate in eq. (4)./

2. From the de Broglie wave length $\lambda = h/p$ we obtain

$$v = \frac{2\pi\hbar}{m\lambda}; \tag{6}$$

this gives $2.5\times10^{-12}c = 7.5\times10^{-2}$ cm/sec for the electron and 4×10^{-5} cm/sec for the neutron. Modern experimental techniques allows one to perform experiments with atomic waves. But for macroscopic bodies the quantum wave lengths are extremely small: for a human being of weight 50 kg moving with the speed of 1 cm/sec we obtain $\lambda \sim 10^{-31}$ cm, i.e. quantum effects in the translational motion are invisible.

3. Average energy of an atom at this temperature, according to the classical Maxwell distribution, is $(3/2)T$ where temperature is expressed in energetic units ($1\,\text{eV} = 11600\,\text{K}$). From here we obtain the average velocity, $v/c \approx 2 \times 10^{-11}$, and the de Broglie wave length $\lambda \approx 6 \times 10^{-5}$ cm. If this

wave length is comparable to the size r_0 of a small cube accomodating just one atom, then the density of the gas is $n \approx r_0^{-3} \approx 0.5 \times 10^{-12}\,\mathrm{cm}^{-3}$. This is a typical atomic density in the traps used for studies of the Bose-Einstein condensate. /This estimate gives only the conditions for the transition from the Maxwell distribution to the Bose-Einstein distribution. At very small temperature, all atoms occupy the ground state available in the trap so that our initial estimate for atomic energy becomes invalid, and typical atomic velocities are determined by the size of the trap in accordance with the uncertainty relation./

4. The potential energy of an electron in the field of the screened center is

$$U(r) = -\frac{Ze^2}{r} e^{-\kappa r}. \tag{7}$$

Consider a circular electron orbit of radius r and speed v. The equilibrium condition for this orbit reads

$$\frac{mv^2}{r} = \frac{Ze^2\kappa}{r}\left(1 + \frac{1}{\kappa r}\right) e^{-\kappa r}, \tag{8}$$

or, applying the quantization rule

$$L^2 = (mvr)^2 = n^2\hbar^2, \tag{9}$$

we find

$$Ze^2 m\kappa \left(1 + \frac{1}{\kappa r}\right) r^2 e^{-\kappa r} = n^2\hbar^2. \tag{10}$$

The left hand side of eq. (10) exponentially falls off for large distances, $\kappa r \gg 1$. Therefore there is no solutions for large values of n, and the number of bound states should be finite. The maximum allowed radius can be found from the maximum of the left hand side, which is given by the positive root of the equation

$$r^2 - \frac{r}{\kappa} - \frac{1}{\kappa^2} = 0 \quad \rightsquigarrow \quad r = \frac{1+\sqrt{5}}{2\kappa}. \tag{11}$$

Of course, one could guess with no calculations that the maximum radius of the orbit should be of the order $r_D = 1/\kappa$. The maximum quantum number, corresponding to the number of levels supported by the screened potential, is now determined from

$$n_{max}^2 = \frac{(3+\sqrt{5})(1+\sqrt{5})}{4} e^{-(1+\sqrt{5})/2}\, \frac{mZe^2}{\kappa\hbar^2} \approx 0.84\, \frac{r_D}{a_B}\, Z, \tag{12}$$

where $a_B/Z = \hbar^2/(me^2 Z)$ is the Bohr radius of the lowest bound orbit in the pure Coulomb potential of the charge Z. Although the semiclassical

Bohr-Sommerfeld quantization usually is not accurate for the lowest orbit, nevertheless we get a reasonable estimate that for a very low value of the Debye radius, $r_D < a_B/Z$ the screened potential does not support bound states at all, $n_{max} < 1$.

The Yukawa-type potential arises in many problems of elementary particle physics when the interaction between two objects of mass M is mediated by the exchange of a *meson* (intermediate particle) of mass μ; the attractive exchange potential in this case can be written (we will come to this later studying relativistic quantum mechanics) as

$$U(r) = -\frac{f^2}{r} e^{-(\mu c/\hbar)r}. \tag{13}$$

Here the squared *coupling constant* f^2 has a dimension [energy×distance], and the role of the Debye radius is played by the *Compton wave length* $\hbar/\mu c$ of the meson. The meson exchange, according to the previous results, does not create a bound state of two particles, if the attraction is too weak, $(f^2/\hbar c) < 1.19(\mu/m)$, where $m = M/2$ is the reduced mass of the interacting particles. This result is quite close to the exact one, $f^2_{crit}/(\hbar c) = 0.84(\mu/m)$ that can be obtained with the aid of a numerical solution of the Schrödinger equation for the Yukawa potential.

5. The relevant transition in the Balmer series of the He^+ ion (Z=2) is between $n = 3$ and $n' = 2$ levels. In the units of $2\pi\hbar c R_H$, where R_H is the Rydberg constant for the hydrogen atom, the transition energy is proportional to $[(1/4) - (1/9)]Z^2 = 5/9$. Meanwhile, the excitation of the hydrogen atom in the lowest excited state ("Lyman-α" transition $n' = 1 \to n = 2$) requires, in the same units, a larger amount of energy 1-(1/4)=(3/4). This is possible only at the expense of the kinetic energy of relative motion. The ratio of the required frequency ω' to the natural frequency ω which would be emitted by the ion at rest should be

$$\frac{\omega'}{\omega} = \frac{3/4}{5/9} = \frac{27}{20}. \tag{14}$$

The transformation of frequencies for the moving ion is given by the Doppler shift for relative velocity v,

$$\frac{\omega'}{\omega} = \sqrt{\frac{1+\beta}{1-\beta}}, \quad \beta = \frac{v}{c}. \tag{15}$$

We determine the velocity necessary for the *resonance* between the emitted and absorbed wave lengths from eqs. (14) and (15):

$$\beta = \frac{\omega'^2 - \omega^2}{\omega'^2 + \omega^2} = \frac{329}{1129} \quad \rightsquigarrow \quad v = 0.29\,c. \tag{16}$$