<u>SOLUTIONS for Homework #2</u>

1. The given wave function can be normalized to the total probability equal to 1,
$$\psi(x) = Ne^{-\lambda|x|}. \tag{1}$$
To get
$$\int_{-\infty}^{\infty} dx\, |\psi(x)|^2 = 2|N|^2 \int_0^{\infty} dx\, e^{-2\lambda x} = 1, \tag{2}$$
we choose
$$N = \sqrt{\lambda}. \tag{3}$$
This state corresponds to the average position $\langle x\rangle = 0$ and the coordinate uncertainty
$$\langle x^2\rangle = \int_{-\infty}^{\infty} dx\, |\psi(x)|^2 x^2 = \frac{1}{2\lambda^2}. \tag{4}$$
The wave function in the momentum representation is given by the Fourier expansion,
$$\phi(k) = \int_{-\infty}^{\infty} dx\, \psi(x) e^{-ikx}, \tag{5}$$
or, after simple calculation,
$$\phi(k) = \sqrt{\lambda}\left(\frac{1}{\lambda+ik} + \frac{1}{\lambda-ik}\right) = \frac{2\lambda^{3/2}}{\lambda^2+k^2}, \tag{6}$$
the *Lorentzian* in momentum space is the Fourier image of the exponential packet in coordinate space. The centroid of $\phi(k)$ is, obviously, at $\langle k\rangle = 0$, while its variance can be defined as
$$\langle k^2\rangle = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, |\phi(k)|^2 k^2 = \lambda^2. \tag{7}$$
Thus, we have
$$\Delta x = \frac{1}{\sqrt{2}\lambda}, \quad \Delta p = \hbar\Delta k = \hbar\lambda, \tag{8}$$
and therefore
$$(\Delta x)(\Delta p) = \frac{\hbar}{\sqrt{2}} > \frac{\hbar}{2}, \tag{9}$$
in accordance with the uncertainty relation.

2. To find the lower bound we assume that the quantum spreading during given time τ is much greater than the original position uncertainty. Then
$$\Delta x \simeq \sqrt{\frac{\hbar\tau}{m}}. \tag{10}$$

For $\tau \approx 10^{10}$ years $\approx 3\times10^{17}$ sec this gives $\Delta x \simeq 6\times10^{8}$ cm for an electron, $\simeq 1.5\times10^{7}$ cm for a proton, $\simeq 3\times10^{-6}$ cm for an object of $m = 1$ g. For the Universe we can make an estimate starting with an equivalent number of protons 10^{-5} cm^{-3} which would give a critical density $\Omega = 1$ of the Universe. This would give, in the volume of $c \times t = 10^{10}$ light years, total mass $M \approx 10^{82} m_e$, or the spreading distance by 41 order of magnitude smaller than for an electron...

3. Let us consider the components $\phi_{1,2}(k)$ of the initial wave function in the momentum representation,

$$\psi_{1,2}(x,0) = \int \frac{dk}{2\pi}\, \phi_{1,2}(k) e^{ikx}. \tag{11}$$

/We always define the Fourier transformation with the factor $(2\pi)^{-d}$, where d is the dimension of space, in the momentum integral $\int d^dk$./ If the energy of a plane wave with wave vector k is $\epsilon(k)$, for example $\hbar^2 k^2/2m$, the time evolution of these components is given by

$$\psi_{1,2}(x,t) = \int \frac{dk}{2\pi}\, \phi_{1,2}(k) e^{ikx-(i/\hbar)\epsilon(k)t}. \tag{12}$$

Therefore we can find the time dependence of the overlap,

$$\gamma(t) = \int dx \left[\int \frac{dk_1}{2\pi}\, \phi_1(k_1) e^{ik_1x-(i/\hbar)\epsilon(k_1)t}\right]^* \left[\int \frac{dk_2}{2\pi}\, \phi_2(k_2) e^{ik_2x-(i/\hbar)\epsilon(k_2)t}\right]. \tag{13}$$

First we notice that the coordinate integral cancels all interference terms except for those with coherent spatial phases, $k_1 = k_2$,

$$\int dx\, e^{-ik_1x+ik_2x} = 2\pi\delta(k_1 - k_2). \tag{14}$$

Now we can take into account this δ-function to integrate over one of the wave vectors and see that the energy dependent phases cancel,

$$\gamma(t) = \int \frac{dk}{2\pi} \phi_1^*(k)\phi_2(k) = \gamma(0). \tag{15}$$

The overlap is constant in time. The reason for this, as can be seen from the derivation, is that only the mutually coherent parts of the packet interfere in infinite space, and they have the same energy dependent phases.

4. In the frame K' the wave function has a standard form being expressed through the coordinate x' with respect to this frame,

$$\Psi'(x',t) = Ae^{ik'x'-i\omega' t}, \tag{16}$$

while in the frame K

$$\Psi(x,t) = Ae^{ikx-i\omega t}. \tag{17}$$

The coordinates are related by the *Galilean transformation*

$$x = x' + ut. \tag{18}$$

Therefore the velocities are related as

$$v = v' + u \quad \rightsquigarrow \quad k = \frac{mv}{\hbar} = \frac{m(v'+u)}{\hbar} = k' + \frac{mu}{\hbar}, \tag{19}$$

and energies as

$$\hbar\omega = \frac{mv^2}{2} = \frac{m(v'+u)^2}{2}. \tag{20}$$

The result can be written as

$$\Psi(x,t) = \Psi'(x',t') \exp\left[\frac{i}{\hbar}\left(mux - \frac{mu^2}{2}t\right)\right]. \tag{21}$$

The result is the product of the function $\Psi'(x',t')$ describing the particle motion in the frame K' and the exponent which characterizes the free motion of this particle together with the frame K' relative to the frame K.

5. Again, see *Problem 3*, it is convenient to choose the normalization in such a way that

$$\int \frac{dk}{2\pi} |\phi(k)|^2 = 1, \quad \rightsquigarrow N^2 = \alpha\sqrt{8\pi}. \tag{22}$$

The wave function in the coordinate representation is given by

$$\Psi(x,0) = N \int \frac{dk}{2\pi} e^{ikx-\alpha^2(k-k_0)^2}. \tag{23}$$

This integral can be easily evaluated by forming the full square in the exponent and using the standard Gaussian integral

$$\int_{-\infty}^{\infty} dz e^{-z^2/2\sigma^2} = \sqrt{2\pi\sigma^2}. \tag{24}$$

We obtain (the normalization has to be correct automatically)

$$\Psi(x,0) = \frac{1}{(2\pi\alpha^2)^{1/4}} e^{ik_0x - x^2/(4\alpha^2)}. \tag{25}$$

This is again the Gaussian packet moving as a whole with momentum $\hbar k_0$.

a. Using these results we find the initial Gaussian probability density

$$\rho(x,0) = |\Psi(x,0)|^2 = \frac{1}{\sqrt{2\pi\alpha^2}} e^{-x^2/(2\alpha^2)} \tag{26}$$

b. According to eq. (26), $\langle x \rangle = 0$ and $\langle x^2 \rangle = (\Delta x)^2 = \alpha^2$. From the original momentum wave function we find

$$\langle k^2 \rangle = N^2 \int \frac{dk}{2\pi} \, e^{-2\alpha^2(k-k_0)^2} k^2 = k_0^2 + \frac{1}{4\alpha^2}. \tag{27}$$

Therefore

$$(\Delta p)^2 = \hbar^2(\langle k^2 \rangle - \langle k \rangle^2) = \frac{\hbar^2}{4\alpha^2}, \tag{28}$$

and the uncertainty product at $t = 0$ is

$$[(\Delta x)(\Delta p)]_0 = \frac{\hbar}{2}. \tag{29}$$

Later in the course we show that this value is the minimum possible for any quantum state of a particle. Thus, the conclusion is that the Gaussian packet *minimizes* the uncertainty relation.

c. Each momentum component of the wave function evolves with time independently acquiring the phase corr to the dispersion law $\epsilon(k)$, see again *Problem 3.* Therefore eq. (23) shows that

$$\Psi(x,t) = N \int \frac{dk}{2\pi} e^{ikx - \alpha^2(k-k_0)^2} e^{-i\hbar(k^2/2m)t}. \tag{30}$$

The integration gives the time-dependent Gaussian packet,

$$\Psi(x,t) = \frac{1}{(2\pi\alpha^2)^{1/4}} \frac{1}{\sqrt{1 + (i\hbar t/2m\alpha^2)}} \exp\left\{ - \frac{(x^2 - 4i\alpha^2 k_0 x + 2i\alpha^2(\hbar k_0^2/m)t}{4\alpha^2[1 + (i\hbar/2m\alpha^2)t]} \right\}, \tag{31}$$

with the coordinate probability density, compare with (26),

$$\rho(x,t) = \frac{1}{\sqrt{2\pi\alpha^2(t)}} \exp\left\{ - \frac{[x - (\hbar k_0/m)t]^2}{2\alpha^2(t)} \right\}, \tag{32}$$

with the effective time-dependent parameter of the packet

$$\alpha^2(t) = \alpha^2 \left[1 + \left(\frac{\hbar t}{2m\alpha^2} \right)^2 \right]. \tag{33}$$

Obviously, the probability density is normalized to 1 for all times, as it should be due to the probability conservation.

d. The center of the packet is moving with the classical speed,

$$\langle x(t)\rangle = \frac{\hbar k_0}{m}\, t \tag{34}$$

(the group velocity of the wave packet equals velocity of the particle). The packet is spreading with the width growing as

$$(\Delta x)_t = \alpha(t) = \alpha\sqrt{1 + (\hbar/2m\alpha^2)^2 t^2}. \tag{35}$$

We see that the effects of the initial size and subsequent quantum spreading add in quadratures. At sufficiently large time, $t \gg m\alpha^2/\hbar$, the width grows linearly,

$$(\Delta x)_t \simeq \frac{\hbar}{2m\alpha}\, t. \tag{36}$$

With the initial momentum uncertainty $\Delta k \sim 1/2\alpha$, the spread of speeds of various Fourier components is $\Delta v \sim \Delta p/m \sim \hbar/2m\alpha$, and the spreading of the packet during time t becomes $\Delta x \sim t\Delta v$, in agreement with (35).

Since the momentum wave function acquires only a time-dependent phase, the momentum probability density $\Phi(p,t)$ does not change with time (each plane wave component propagates independently without changing its amplitude). This means that $\langle p\rangle$, $\langle p^2\rangle$ and Δp stay constant. The difference between the momentum and coordinate spreads is due to the fact that the momentum is constant of motion for free particles. The uncertainty relation now takes the form

$$(\Delta x)(\Delta p) = \hbar\frac{\alpha(t)}{2\alpha} = \sqrt{1 + \left(\frac{\hbar t}{2m\alpha^2}\right)^2}\,\frac{\hbar}{2}. \tag{37}$$