Lecture 25

Chapter 31
Induction and Inductance
Review

- Magnetic flux
  \[ \Phi_B = \int B \cdot dA \]

- Faraday’s law
  \[ E = -\frac{d\Phi_B}{dt} \]

- Lenz’s law – induced emf gives rise to a current whose \( B \) field opposes the change in flux that produced it

- Changing \( B \) field produces an \( E \) field

- Restate Faraday’s law
  \[ \oint \varepsilon \cdot ds = -\frac{d\Phi_B}{dt} \]

- Inductance, \( L \) defined
  \[ L = \frac{N\Phi_B}{i} \]
Review

- Inductor – device produces known $B$ field
- Solenoid is an inductor with inductance per unit length of $\frac{L}{l} = \mu_0 n^2 A$
- Self-induce emf, $E_L$ appears in any coil in which the current is changing
  \[ E_L = -L \frac{di}{dt} \]
- Direction of $E_L$ follows Lenz’s law and opposes the change in current
Review

- RL circuit – resistor and inductor in series
- Time dependence on current in RL circuit
- Initially inductor acts to oppose changes in current through it
- Long time later, inductor acts like simple wire

- Rise of current
  \[ i = \frac{E}{R} \left(1 - e^{-t/\tau_L}\right) \]

- Decay of current
  \[ i = \frac{E}{R} e^{-t/\tau_L} = i_0 e^{-t/\tau_L} \]

- Inductive time constant
  \[ \tau_L = \frac{L}{R} \]
Inductance (39)

- Mutual induction – current in one coil induces emf in other coil
- Distinguish from self-induction
- Mutual inductance, $M_{21}$ of coil 2 with respect to coil 1 is

$$M_{21} = \frac{N_2 \Phi_{21}}{i_1}$$
Inductance (40)

- Faraday’s law
  \[ E_2 = -N_2 \frac{d\Phi_{21}}{dt} \]

- Rearrange equation
  \[ M_{21} i_1 = N_2 \Phi_{21} \]

- Vary \( i_1 \) with time
  \[ M_{21} \frac{di_1}{dt} = N_2 \frac{d\Phi_{21}}{dt} \]

- Induced emf in coil 2 due to \( i_1 \) in coil 1 is
  \[ E_2 = -M_{21} \frac{di_1}{dt} \]

- Obeyes Lenz’s law (minus sign)
Inductance (41)

- Reverse roles of coils
- What is induced emf in coil 1 from a changing current in coil 2?
- Same game as before

\[ M_{12} = \frac{N_1 \Phi_{12}}{i_2} \]

\[ E_1 = -M_{12} \frac{di_2}{dt} \]
Inductance (42)

• The mutual inductance terms are equal

\[ M_{21} = M_{12} = M \]

• Rewrite emfs as

\[ E_2 = -M \frac{di_1}{dt} \quad E_1 = -M \frac{di_2}{dt} \]

• Notice same form as self-induced emf

\[ E_L = -L \frac{di}{dt} \quad L = \frac{N \Phi_B}{i} \]
Inductance (43)

- **Generators** – convert mechanical energy to electrical energy
- **External agent rotates loop of wire in** $B$ **field**
  - Hydroelectric plant
  - Coal burning plant
- **Changing** $\Phi_B$ **induces an emf and current in an external circuit**

![Diagram showing a coil with a rotating loop of wire, a magnetic field, sliding contacts, and a load resistor $R$.]
Inductance (44)

- **Alternating current (ac) generator**
  - Ends of wire loop are attached to slip rings which rotate with loop
  - Stationary metal brushes are in contact with slip rings and connected to external circuit
  - emf and current in circuit alternate in time
Inductance (45)

- Calculate emf for generator with N turns of area A and rotating with constant angular velocity, $\omega$
- Magnetic flux is

$$\Phi_B = \int \vec{B} \cdot d\vec{A} = BA \cos \theta$$

- Relate angular displacement to angular velocity

$$\theta = \omega t$$

- Flux through one loop is

$$\Phi_B = BA \cos \omega t$$
Inductance (46)

• Faraday’s law says
  \[ E = -N \frac{d\Phi_B}{dt} \]
  \[ \Phi_B = BA \cos \omega t \]
• Substitute
  \[ E = -NBA \frac{d}{dt} (\cos \omega t) \]
  \[ E = NBA \omega \sin \omega t \]

• Maximum emf is when \( \omega t = 90 \) or 270 degrees
  \[ E_{\text{max}} = NBA \omega \]
• Emf is 0 when \( \omega t = 0 \) or 180 degrees
Inductance (47)

• Direct current (dc) generator
  – Ends of loop are connected to a single split ring
  – Metal brush contacts to split ring reverse their roles every half cycle
  – Polarity of induced emf reverses but polarity of split ring remains the same

• Not suitable for most applications
  – Can use to charge batteries

• Commercial dc gen. use out of phase coils
Inductance (48)

- **Motors** – converts electrical energy to mechanical energy
  - Generator run in reverse
  - Current is supplied to loop and the torque acting on the current-carrying loop causes it to rotate
  - Do mechanical work by using the rotating armature
  - As loop rotates, changing $B$ field induces an emf
  - Induced emf (back emf) reduces the current in the loop – remember Lenz’s law
  - Power requirements are greater for starting a motor and for running it under heavy loads
Inductance (49)

• Instead of a loop of wire, what happens when a bulk piece of metal moves through a $B$ field?
• Free electrons in metal move in circles as if caught in a whirlpool called eddy currents
• A metal plate swinging through a $B$ field will generate eddy currents
Inductance (50)

- Eddy currents will oppose the change that caused them – Lenz’s law
- Induced eddy currents will always produce a retarding force when plate enters or leaves $B$ field causing the plate to come to rest
- Cutting slots in metal plate will greatly reduce the eddy currents
Inductance (51)

• Induction and eddy currents are used for braking systems on some subways and rapid transit cars
• Moving vehicle has electromagnet (e.g. solenoid) which is positioned near steel rails
• Current in electromagnet generates $B$ field
• Relative motion of $B$ field to rails induces eddy currents in rails
• Direction of eddy currents produce a drag force on the moving vehicle
• Eddy currents decrease steadily as car slows giving a smooth stop
Inductance (52)

- Eddy currents often undesirable since they dissipate energy in form of heat
- Moving conducting parts often laminated
  - Build up several thin layers separated by nonconducting material
  - Layered structure confines eddy currents to individual layers
- Used in transformers and motors to minimize eddy currents and improve efficiency
Inductance (53)

- How much energy is stored in a $B$ field?
- Conservation of energy expressed in loop rule

\[
E = L \frac{di}{dt} + iR
\]

- Multiply each side by $i$

\[
Ei = Li \frac{di}{dt} + i^2 R
\]

- $Ei$ is rate at which emf device delivers energy to rest of circuit
- $i^2 R$ is rate at which energy appears as thermal energy in resistor
Inductance (54)

- Middle term represents the rate $dU_B/dt$ at which energy is stored in the $B$ field

$$E_i = Li \frac{di}{dt} + i^2 R$$

$$\frac{dU_B}{dt} = Li \frac{di}{dt}$$

$$dU_B = Lidi$$

- Integrating gives

$$\int_0^i dU_B = \int_0^0 Lidi$$

- Energy stored in $B$ field

$$U_B = \frac{1}{2} Li^2$$

- Similar to $U_E$

$$U_E = \frac{1}{2} \frac{q^2}{C}$$
Inductance (55)

- What is the energy density of $B$ field?
- Energy density, $u_B$, is energy per unit volume
- Volume is area x length

$$u_B = \frac{U_B}{Al}$$

$$U_B = \frac{1}{2} Li^2$$

Substituting $U_B$ gives

$$u_B = \frac{Li^2}{l2A}$$

For a solenoid

$$\frac{L}{l} = \mu_0 n^2 A$$

Energy density is

$$u_B = \frac{1}{2} \mu_0 n^2 i^2$$
Inductance (56)

- Substituting $B$ gives magnetic energy density

$$u_B = \frac{1}{2} \mu_0 n^2 i^2$$

- Remember $B$ field from a solenoid is

$$B = \mu_0 in$$

- Similar to electric energy density

$$u_E = \frac{1}{2} \varepsilon_0 E^2$$
Inductance (57)

• Place coil C at center of long solenoid which has a steadily decreasing current. What is the magnitude of the induced emf in coil C?
• Solenoid generates uniform $B$ field of

$$B = \mu_0 in$$

• Current is decreasing so $B$ field decreases
Inductance (58)

• Since $B$ field decreases the flux decreases and an emf is induced in coil $C$ (Faraday’s law)

\[
\Phi_B = \int \vec{B} \cdot d\vec{A} \quad \text{E} = -N \frac{d\Phi_B}{dt}
\]

• The current is decreasing at a steady rate so flux also decreases at steady rate and write

\[
\frac{d\Phi_B}{dt} = \frac{\Delta \Phi_B}{\Delta t} = \frac{\Phi_{B,f} - \Phi_{B,i}}{\Delta t}
\]
Inductance (59)

• Need to find initial and final flux
• Current decreases to zero so final flux = 0
• Initial flux

\[ \Phi_{B,i} = \int \vec{B} \cdot d\vec{A} \]

• Solenoid has uniform \( B \) field and it is directed \( \perp \) to area of coil C

\[ \Phi_{B,i} = BA = \mu_0 i nA \]
Inductance (60)

- Substituting into equation for emf

\[ E = -N \frac{\Phi_{B,f} - \Phi_{B,i}}{\Delta t} = N \frac{0 - \mu_0 i n_S A_C}{\Delta t} \]

- Only want magnitude of emf so can ignore minus sign
- \( n_S \) is \( N/L \) of solenoid but \( A_C \) is area of coil \( C \)