ELECTROSTATICS

ELECTRIC CHARGE

- Electric charge is a property of atomic particles, the electron and the proton, which make up atoms (together with neutrons).
- The standard unit of charge is the Coulomb (C).
- Electric charge and mass of particles

Particle	Electric Charge	Mass
Electron	$-e = -1.6 \cdot 10^{-19}$ C	$m_e = 9.11 \cdot 10^{-31}$ kg
Proton	$+e = 1.6 \cdot 10^{-19}$ C	$m_p = 1.672 \cdot 10^{-27}$ kg
Neutron	0	$m_n = 1.674 \cdot 10^{-27}$ kg

- Law of charges: *Like charges repel, and unlike charges attract.*
- An electric charge q is a charge which is an integer multiple of the fundamental charge constant $e = 1.6 \cdot 10^{-19}$ C, $q = n\,e$. Electric charge is quantized.
- The *net charge* of an object is the difference between the number of protons and electrons in it times the elementary charge constant.
- Law of conservation of net charge: *The net charge of an isolated system remains constant.*
- Electric charge transfer is a transfer of electrons.

  Charging positively: Removal of electrons from an object

  Charging negatively: Addition of electrons to an object

ELECTRIC FORCE

- The mutual electrostatic forces on two point charges are equal and opposite, pointing to (away from) the other particle for unlike (like) charges.
- Coulomb's Law

  The electrostatic force between two charges q_1 and q_2 separated by a distance r is:

$$
\begin{aligned}
F &= k\frac{q_1\,q_2}{r^2} \\
k &= 8.99 \cdot 10^9\ \mathrm{N\,m^2/C^2}
\end{aligned}
$$

- Charges interact pairwise via Coulomb force. The superposition principle is valid:

  The net force acting on any charge is the vector sum of the forces due to each of the remaining charges in a given distribution.

ELECTRIC FIELD

- *Test charge* = charge which feels the force of other charges, but exerts no force on them. (mathematical construction)
- *Electric field,* $\vec{E}$ = force per unit test charge: $\vec{E} = \vec{F}/q_0$.
  SI-unit of the $\vec{E}$-field: N/C.
- Electric field of a point charge:
  — Force between two point charges: $F = kQq_0/r^2$.
  — E-field felt by test charge q_0 at r due to the presence of Q is then: $E = F/q_0 = kQ/r^2$.
  — Direction of $\vec{E}$ = direction of $\vec{F}$.
  — Unit positive test charge would be attracted to a negative charge. $\vec{E}$-field points towards a negative point charge and away from a positive point charge.
- Superposition of electric fields: $\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + ...,\quad \vec{E} = \vec{F}/q_0 \quad \rightarrow \vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3 + ...$
- Rules for electrical field lines:
  — The lines are directed pointing away from the positive and towards the negative charges.
  — At any given point in space, the tangent to the line is the direction of the $\vec{E}$-field at that point.
  — The number of lines drawn to or from a charge is proportional to the magnitude of the charge.
- Consequences of these rules:
  — In the immediate vicinity of a point charge, field lines are radially directed.
  — Field lines do not intersect in a charge-free region.
  — Field lines do not begin or end in a charge-free region.
- Density of field lines (number of field lines per unit area) is proportional to the $\vec{E}$-field; and by convention, the total number of field lines is proportional to the charge q.

ELECTRIC DIPOLE

- An arrangement of two equal but opposite charges q separated by a fixed distance d is called a dipole. In a uniform field E, a fixed dipole is subject to a torque: $\tau = qdEsin\theta$. θ is the angle between the dipole direction and the field. $p = qd$ is the dipole moment.
- The field of a dipole along the dipole axis at large distances ($z >> d$) is: $E = 2kp/z^3$
- The field of a dipole along an axis perpendicular to the dipole axis at large distances ($x >> d$) is: $E = kp/x^3$

CONTINUOUS CHARGE DISTRIBUTIONS

- The linear charge density is defined as $\lambda = Q/L$
  The surface charge density is defined as $\sigma = Q/A$
  The volume charge density is defined as $\rho = Q/V$
- The electric field along the z-axis of a charged ring is: $E = kqz/(z^2 + R^2)^{3/2}$
- The electric field along the z-axis of a charged disk is: $E = \sigma/2\epsilon_0(1 - z/\sqrt{z^2 + R^2})$

ELECTRIC FLUX AND GAUSS'S LAW

- The electric flux through a flat surface is given by: $\Phi = \vec{E}\vec{A}$, where the direction of the area is given by the normal vector of the area which is perpendicular to the surface. The magnitude of the flux is then given by $\Phi = EA\cos\theta$, where θ is the angle between the electric field and the normal vector.
- The electric flux Φ through a closed surface (Gaussian surface) is defined as: $\Phi = \oint \vec{E}d\vec{A}$.
- Flux into an enclosed surface is negative and flux out of an enclosed surface is positive.
- Gauss's Law: The net charge enclosed by a closed surface is equal to the flux through this surface times ϵ_0: $q_{enc} = \epsilon_0 \Phi$.
- The potential energy of a point charge q in the field of another point charge Q is $U = kqQ/r$.

FIELD OF A CHARGED CONDUCTOR

- The field outside of a charged sphere will be the same as the field of a point charge at the center of the sphere. The field inside of a charged conductor is zero.
- The net charge on an isolated conductor is always on the surface of the conductor. The electric field at the surface of a charged conductor is perpendicular to the surface.
- On irregularly shaped charged conductors, the charges will accumulate more at sharp points.

E-FIELDS OF CONTINUOUS CHARGE DISTRIBUTIONS

The following formulas are valid for infinitely long lines, infinitely large areas, and spheres of radius R of uniformly charged materials:

- Line of charge: $E = \lambda/(2\pi\epsilon_0 r)$
- Surface of an infinitely thick sheet of conductor: $E = \sigma/\epsilon_0$
- Sheet of a non-conductor: $E = \sigma/(2\epsilon_0)$
- Sheet of conductor $E = \sigma_1/\epsilon_0$ (The total charge spreads out over 2 surfaces, σ_1 corresponds to the charge on one surface)
- Field between a positively and negatively charged conductor: $E = 2\sigma_1/\epsilon_0 = \sigma/\epsilon$ (σ now corresponds to the total charge, which is accumulated on the inside of the conductors)
- The field outside ($r > R$) a sphere: $E = kq/r^2$
- The field inside ($r < R$) a conducting sphere: $E = 0$
- The field inside ($r < R$) a nonconducting sphere: $E = kqr/R^3$

ELECTRIC POTENTIAL ENERGY

- The electrostatic potential energy, U is defined as the difference in potential energy of a charge q at two different points i and f: $\Delta U = U_f - U_i$.
- It is equal to the work W done **by** the electrostatic force in transporting this charge from the initial position i to the final position f: $\Delta U = -W$.
- The reference point of $U = 0$ is at $i = \infty$ and thus the electrical potential energy at point f is: $U = -W$.

ELECTROSTATIC POTENTIAL

- Definition: Electrostatic potential difference: $\Delta V = V_f - V_i = \Delta U/q = -W/q$
- With $U_i = U_\infty = 0$ the potential is: $V = -W_\infty/q$.

EQUIPOTENTIAL SURFACE AND RELATION BETWEEN V AND E

- Definition of equipotential surfaces: Surfaces of constant potential.
- The electric field lines are always perpendicular to the equipotential surfaces.
- Metal surfaces are equipotential surfaces.
- From the relation $dW = \vec{F}d\vec{s}$ the potential between two points separated by a distance s can be calculated from the electric field: $V = -\int_i^f \vec{E}d\vec{s}$
- The component of the electric field in a given direction can be calculated from the potential: $E = -\partial V/\partial s$

POTENTIAL OF POINT CHARGES

- The potential of a single point charge is given by: $V = kq/r$. The potential at infinity is assumed to be zero and this equation is valid including the sign when the sign of the charge is also included.
- Again, the superposition principle is valid, the potential of several point charges is equal to the sum of the individual potentials: $V = V_1 + V_2 + V_3 + ... + V_n = \sum_{i=0}^n V_i = k\sum_{i=0}^n (q_i/r_i)$
- A positive charge accelerates from a region of higher potential to a region of lower potential.
- The potential due to a dipol at large distances r is given by: $V = kp\cos\theta/r^2$, where p is the dipole moment and θ is the angle between the dipole axis (pointing from the negative to the positive charge of the dipole) and $\vec{r}$.

CAPACITORS

- The potential difference between two charged plates A and B is related to the E-field via: $V = Ed$.
- Definition: The *capacitance* C is a proportionality constant that relates q to V: $q = CV$ or $C = q/V$, SI Unit: $1Farad = 1F = 1C/V$
- Parallel capacitor: (q is the total charge on one plate): $4\pi kq = EA = (V/d)A \Rightarrow$ $C = q/V = A/(4\pi kd) = \epsilon_0 A/d$
- Cylindrical capacitor: $C = 2\pi\epsilon_0 L/(ln(b/a)$, where a and b correspond to the radius of the inner and outer plate, respectively.
- Spherical capacitor: $C = 4\pi\epsilon_0 ab/(b-a)$, where a and b correspond to the radius of the inner and outer spheres, respectively.
- Isolated Sphere: $C = 4\pi\epsilon_0 R = kR$.

CAPACITORS IN PARALLEL

- The sides of two capacitors connected together are on the same potential: $V_1 = V_2 = V$. The charges on each capacitor are: $q_1 = C_1 V$ and $q_2 = C_2 V$. The total charge is : $q = q_1 + q_2$.
  Therefore, the capacitance of the equivalent capacitor is:
  $C_p = q/V = (q_1 + q_2)/V = C_1 + C_2 \Rightarrow C_p = C_1 + C_2$, or for n capacitors in parallel: $C = \sum_i^n C_i$.

CAPACITORS IN SERIES

- If the capacitors are connected in series the two connected plates in the middle have zero net charge. Therefore the charges on both capacitors are equal: $q_1 = q_2 = q$.
  The total voltage is then $q/C_s = V = V_1 + V_2 = q_1/C_1 + q_2/C_2 = q/C_1 + q/C_2 \Rightarrow$
  $1/C_s = 1/C_1 + 1/C_2$, or for n capacitors in series: $1/C = \sum_i^n 1/C_i$.

ENERGY STORED IN A CAPACITOR

- Potential energy differences are created by transporting charge from one plate to the other.
  $W = Vq/2$ with $q = CV \Rightarrow W = q^2/(2C) = CV^2/2$.

DIELECTRICS

- Dielectrics are materials composed of permanent dipoles which can be reoriented but cannot move. The dipoles align and partially compensate the external field. With fixed charges in the capacitor, the E-field between the plates is reduced with the dielectric present as compared to without.
- Definition: *Dielectric constant*, κ, $\kappa = C_\kappa/C_0 = E_0/E_\kappa = V_0/V_\kappa$ where E_0 is the electric field without the dielectric present, and E_κ is the field with it present.
- With the permittivity of the dielectric material $\epsilon = \epsilon_0\,\kappa$: $C_\kappa = \kappa\, C_0 = \epsilon_0\,\kappa A/d = \epsilon\, A/d$.

CURRENT AND CIRCUITS

ELECTRIC CURRENT

- Amount of current = amount of charge that passes per time unit through an area perpendicular to the flow: $i = dq/dt$, Unit: C/s = A (Ampere).
- The current density is defined as $J = i/A$, where A is the cross sectional area.
- Convention: Direction of current = direction of positive charge flow. Since e^- are the moving charges, the defined direction of the current is opposite to the direction of the physical current.
- The current is related to the electron drift velocity which is amazingly small: $v_d = J/(ne)$
  It is typically $/sim$ mm/s. ne is the charge carrier density.
- Why does current flow instantaneously? Because of the E-field which moves with speed of light, which causes all electrons in wire to drift at the same time.

RESISTANCES

- If one connects a wire between both terminals of a battery, a current flows. The resistance is defined as $R = V/i$ (Ohm's law), Unit: Ω.
- Physical reason for resistance: Scattering of the conduction electrons off obstacles in the conductor. Ohm's law is not generally valid, but it is a good empirical rule for most systems.
- The resistivity is defined as $\rho = E/J$.
- The overall resistance of a wire should be proportional to its length and inversely proportional to its cross sectional area The proportionality constant is called resistivity, ρ.
  $R = \rho\, L/A \Rightarrow \rho = R\, A/L$, Unit: Ωm
- The resistivities are temperature dependent due to thermal vibrations in the material:
  $\rho - \rho_0 = \rho_0 \alpha (T - T_0)$.
- Resistivities are usually tabulated at room temperature, 20 °C.
- The conductivity is: $\sigma = 1/\rho$.

POWER DISSIPATION IN A SIMPLE CIRCUIT

- Potential drop across resistor (Ohm's law): $V = i\, R$.
- Since charge is transported from the positive end of the load resistor to the negative end across a potential V, it loses potential energy $PE_{elec} = V\,\Delta q = V\, i\,\Delta t$. This potential energy is converted into some other form of energy, here: heat (Joule heating).
- The power dissipated is the change in potential energy (work) per time unit:
  $P = \Delta W/\Delta t = V\, i\,\Delta t/\Delta t = V\, i$.
- Using $V = R\, i$, we can also write for the power $P = i^2\, R = V^2/R$.

BATTERIES

- Batteries supply a potential difference which is called an *electromotive force* (emf). This emf is defined by the work dW done on a charge dq: $\mathcal{E} = dW/dq$.
- Real batteries have an internal resistance r when current is drawn from the battery. This internal resistance reduces the nominal *emf* according to Ohm's law to the *terminal voltage* (TV) which is given by $TV = \mathcal{E} - ir$.
- Ideal batteries do not have any internal resistances.

KIRCHHOFF'S RULES

- *Kirchhoff's first rule: (Junction Rule)* The sum of the currents flowing into a junction is equal to the sum of the currents flowing out of the junction (conservation of charge).
- *Kirchhoff's second rule: (Loop Rule)* The sum of the potential drops is equal to the sum of the potential rises within a closed loop (conservation of energy).
- *Resistance rule:* For a move through a resistor in the direction of the current, the change in potential is $-iR$, in the opposite direction it is $+iR$.
  emf rule: For a move through an ideal emf device in the direction of the emf arrow, the change in potential is $+\mathcal{E}$, in the opposite direction it is $-\mathcal{E}$.

RESISTORS IN SERIES AND IN PARALLEL

- For two resistors in series the current that flows through the circuit is the same everywhere and the voltage drops across the resistors add up:
  $\mathcal{E} = V_1 + V_2 = i_1\,R_1 + i_2\,R_2 = i\,R_1 + i\,R_2 = i\,(R_1 + R_2) = i\,R_s \Rightarrow R_s = R_1 + R_2$
  or for n resistors in series: $R_s = \sum_i^n R_i$.
- For two parallel resistors the potential drop across each resistor is constant: $\mathcal{E} = V_1 = V_2$, and the current that flow through the individual resistors add up to the total current:
  $i = i_1 + i_2 = V_1/R_1 + V_2/R_2 = \mathcal{E}/R_1 + \mathcal{E}/R_2 = \mathcal{E}\,(1/R_1 + 1/R_2) = \mathcal{E}/R_p \Rightarrow 1/R_p = 1/R_1 + 1/R_2$
  or for n resistors in parallel: $1/R_p = \sum_i^n 1/R_i$.

CHARGING AND DISCHARGING A CAPACITOR

- It takes time to put charges onto a capacitor. Initally it is fast, but as more charges are on the capacitor it becomes harder: $q = C\mathcal{E}(1 - e^{-t/RC})$, $i = (\mathcal{E}/R)e^{-t/RC}$. RC is called the time constant of the circuit.
- Discharging the capacitor is again fast at the beginning, getting slower the fewer charges are left on the capacitor: $q = q_0 e^{-t/RC}$, $i = -(q_0/RC)e^{-t/RC}$.

MAGNETISM

MAGNETIC FIELD

- Magnets have two *poles*, named North (N) and South (S). Magnets have *magnetic field lines* extending from the North pole to the South pole. Two like poles repel, two unlike poles attract each other.
- The magnetic field exerts a force on a **moving** charge: $\vec{F}_B = q\vec{v} \times \vec{B}$. The magnitude of this force is $F_B = |q|vB\sin\Phi$, where Φ is the angle between the directions of the velocity and the field.
- The direction of the $\vec{B}$-field is given by a Right-hand-rule:
  When the fingers sweep $\vec{v}$ into $\vec{B}$, the thumb points for positive charges in the direction of the force $\vec{F}_B$. Units of B: 1 $Tesla$ $(T) = Ns/Cm = Vs/m^2$ Other unit: 1 $Gauss(G) : 1G = 10^{-4}T$

TRAJECTORY OF CHARGES IN CONSTANT B-FIELDS

- The magnetic force F_B on a charge moving perpendicular to a magnetic field causes a circular motion. It is a centripetal force: $F_c = m\,v^2/r$. Therefore: $F_m = F_c \Rightarrow q\,v\,B = m\,v^2/r$.
- Solving for the radius of the orbit: $r = m\,v/(q\,B)$. The frequency is then $f_{osc} = 1/T = q\,B/(2\pi\,m)$.

HALL EFFECT

- Moving electrons in a wire (= current) can be deflected by a magnetic field which produces a (Hall) potential difference. In the equilibrium ($F_E = F_B$) it is possible to measure the charge density $n = Bi/(Vle)$, where l is the thickness of the strip of wire.

FORCE ON A CURRENT

- Current consists of moving charges. The magnitude of the magnetic force on a wire of length L with current i due to a magnetic field B is given by $\vec{F}_B = i\vec{L} \times \vec{B}$, $F_B = iLB\sin\theta$, where θ is the angle between the wire (current) and the magnetic field.

TORQUE ON A CURRENT CARRYING COIL

- Assume a coil in a magnetic field $\vec{B}$. For one loop of the coil: The forces on the current in pivoted part of the loop are equal and in opposite directions ($\vec{B} \perp \vec{i}$) and thus there is no net torque.
- The force on the current I in the non-pivoted side of the loop (width a) is $F = iaB$.
- The torque $\vec{\tau}$ on one side of the loop is $\tau = (b/2)F\sin\theta$,
  where θ is the angle between the normal to the loop and the magnetic field.
- Thus, the total torque on the loop (the two opposite sides add up) is
  $\tau = (b/2)iaB\sin\theta + (b/2)iaB\sin\theta = iAB\sin\theta$ where the area of the loop is $A = ab$.
- When the coil consists of N loops the total torque is $\tau = NiAB\sin\theta$.
- The magnetic moment of the coil is defined as: $\mu = NiA$ and thus the torque is given by $\vec{\tau} = \vec{\mu} \times \vec{B}$.

MAGNETIC FIELD DUE TO A CURRENT

- The magnetic field $d\vec{B}$ at a distance r from a wire due to a current i inside a piece $d\vec{s}$ of the wire is given by Bio-Savart's law: $d\vec{B} = \mu_0 i d\vec{s} \times \vec{r}/(4\pi r^3)$
  The constant: $\mu_0 = 4\pi \cdot 10^{-7}\, Tm/A$ is called the permeability of free space.
- Magnetic field of a long straight wire at a distance r from the wire: $B = \mu_0 i/(2\pi r)$
- Magnetic field inside of a long straight wire with radius R at a distance r from the center: $B = \mu_0 ir/(2\pi R^2)$
- Magnetic field at a center of a circular arc: $B = \mu_0 i\Phi/(4\pi R)$, where Φ is measured in radians.
  Thus the magnetic field at the center of a current loop is: $B = \mu_0 i/(2R)$
- Magnetic field of a coil along the symmetry axis z through the center: $B = \mu_0 \vec{\mu}/(2\pi z^3)$,
  where $\mu = NiA$ is the magnetic dipole moment of the coil and N is the number of turns in the coil.
- Right Hand Rule: The direction of the field curles around with the fingers of the right hand when the thumb points in the direction of the current.

FORCE BETWEEN TWO WIRES

- The force on a wire which carries a current i_1 due to the magnetic field of another (parallel) wire with current i_2 ($B = \mu_0 i_2/(2\pi d)$) is given by: $F = \mu_0 L i_1 i_2/(2\pi d)$.
  d is the distance between the two wires and L is the length of the wires.
- If the current in both wires are in the same direction, the force is attractive, and if the currents are in opposite directions the force is repulsive.

MAGNETIC FIELD OF A SOLENOID AND TOROID

- A solenoid is a long straight coil of tightly wound wire. The magnetic field inside the solenoid is directed along the center of the coil: $B = \mu_0 in$, where n is the number of turns per unit length. Thus, if N is the total number of turns of the solenoid of length L, then $n = N/L$.
- A toroid is a solenoid bent into the shape of a doughnut. The magnetic field at the center of the toroid is: $B = \mu_0 iN/(2\pi r)$, where N is the total number of turns and r is the radius of the toroid.

AMPERE'S LAW

- For certain situations which involve symmetries it is easier to use Ampere's law rather than Biot-Savart's law. The integral of the scalar product of the magnetic field $\vec{B}$ and pathsegment $d\vec{s}$ over a closed imaginary loop is proportional to the enclosed current: $\oint \vec{B} d\vec{s} = \mu_0 i_{enc}$.

INDUCTION

- A current I generates a magnetic field B. Can a magnetic field also generate a current? Yes and No. A constant (in time) magnetic field does not generate a current, but changes in the field do.
- Faraday's observations: An emf can be generated in a loop of wire by:
  (i) holding it close to a coil (solenoid) and changing the current in the coil.
  (ii) keeping the current in the coil steady, but moving the coil relative to the loop.
  (iii) moving a permanent magnet in or out of the loop.
  (iv) rotating the loop in a steady magnetic field.
  (v) changing the shape of the loop in the field.
- The Magnetic flux is defined as $\Phi_B = \int \vec{B} d\vec{A}$. If B is constant over the area, then the flux is given by $\Phi_B = BA\cos\theta$. θ is the angle between $\vec{B}$ and $\vec{A}$, the "normal" to the surface.
- A change in the magnetic flux produces a potential difference (and via Ohm's law a current) in a coil: $\mathcal{E} = -Nd\Phi_B/dt$.
  Φ is the flux through the coil and N is the number of turns of the coil.
- Thus, if B is constant within the coil: $\mathcal{E} = -Nd(BA\cos\theta)/dt$. In most cases, only one variable depends on the time:
  (1) A, θ constant: $\mathcal{E} = -NA\cos\theta dB/dt$
  (2) B, θ constant: $\mathcal{E} = -NB\cos\theta dA/dt$
  (3) A, B constant: $\mathcal{E} = -NABd(\cos\theta)/dt$
- Why the negative sign? **Lenz's law:** An induced emf gives rise to a current whose magnetic field opposes the change in flux that produced it.
- The magnitude of the emf of a moving conductor in a perpendicular magnetic field is given by: $\mathcal{E} = BLv$, where L is the length of the conductor and v is the velocity (perpendicular to the field) of the conductor ($v = dx/dt$).
- Faraday's law of induction can also be expressed in terms of an electric field: $\oint \vec{E} d\vec{s} = -d\Phi_B/dt$.

INDUCTANCE AND SELFINDUCTION

- The inductance is defined as $L = N\Phi/i$, Units: 1 henry = 1 H = 1 Vs/A.
- The inductance per unit length of a solenoid is $L/l = \mu_0 n^2 A$.
- A changing current in a coil generates a self-induced emf in the coil: $\mathcal{E}_L = -Ldi/dt$.
- An RL-circuit consists of an inductor and a resistor in series: $Ldi/dt + Ri = \mathcal{E}$. The current rises according to $i = \mathcal{E}/R(1 - exp(-t/\tau_l))$ where $\tau_L = L/R$ is the inductive time constant. After a connection is broken the current decreases as $i = i_0 exp(-t/\tau_l)$.
- The energy stored in an inductor is given by $U_B = Li^2/2$.
  The energy density of a coil is given by $u_B = B^2/(2\mu_0)$.
- The mutual inductance between two coils is defined as M and $E_2 = -Mdi_1/dt$ and $E_1 = -Mdi_2/dt$.

MAGNETIC MATERIALS

- Microscopic origin: Atoms have small magnetic moments (elementary magnetic dipoles). In ferromagnetic materials (iron, nickel ...) these magnetic moments interact so strongly that the dipoles spontaneously align. Thus the material forms one large magnetic dipole → magnet.
- Inserting materials into a magnetic field can polarize the material and create a magnetic dipole moment. The total magnetic field can then be much stronger as it is the sum of the external field and the magnetisation of the ferromagnetic material: $B = B_{ext} + B_M$.

AMPERE-MAXWELL'S LAW AND DISPLACEMENT CURRENT

- Just as a changing magnetic flux creates an electric field (Faraday's law, see page 10 of the lecture notes) a changing electric flux can create a magnetic field: $\oint \vec{B} d\vec{s} = \mu_0 \epsilon_0 d\Phi_E/dt$ (Maxwell's law).
- Thus, a magnetic field can be created by a current or by a changing electric field, which combines Ampere's and Maxwell's law: $\oint \vec{B} d\vec{s} = \mu_0 \epsilon_0 d\Phi_E/dt + \mu_0 i_{enc}$.
- A displacement current can be defined as $i_d = \epsilon_0 d\Phi_E/dt$, and then the Ampere-Maxwell law can be written as $\oint \vec{B} d\vec{s} = \mu_0 i$, with $i = i_{enc} + i_d$.

GAUSS' LAW FOR MAGNETIC FIELDS

- In analogy to Gauss' law for electric fields, it also can be written for magnetic fields: $\Phi_B = \oint \vec{B} d\vec{A} = 0$.
- This means that the net magnetic flux through an enclosed surface is always zero, or: A magnetic monopole does not exist.

MAXWELL'S EQUATIONS

- The basic laws combining electricity and magnetism are Maxwell's equations:

$\oint \vec{E} d\vec{A} = q/\epsilon_0$	$\oint \vec{B} d\vec{A} = 0$
$\oint \vec{E} d\vec{s} = -d\Phi_B/dt$	$\oint \vec{B} d\vec{s} = \mu_0 \epsilon_0 d\Phi_E/dt + \mu_0 i_{enc}$

LC OSCILLATOR

- The total energy stored in an LC oscillator is the sum of the energy stored in the capacitor $U_E = q^2/(2C)$ and the energy stored in the inductor $U_B = i^2 L/2$.
- The total energy is conserved and thus $dU/dt = 0$.
- The solution to the resulting differential equation is: $q = Q cos(\omega t + \phi)$ and $i = I sin(\omega t + \phi)$, where $I = \omega Q$ and $\omega = 1/\sqrt{LC}$.
- The total energy is $U = Q^2/(2C)$, $U_E = Q^2/(2C)\cos^2(\omega t + \phi)$ and $U_B = Q^2/(2C)\sin^2(\omega t + \phi)$. circuit.

RLC OSCILLATOR

- Now energy is dissipated in the resistor and thus $dU/dt = -i^2R$.
  The solution is now: $q = Qe^{-Rt/2L}\cos(\omega' t + \phi)$ with $\omega' = \sqrt{\omega^2 - (R/2L)^2}$.
  The total energy decreases as $U = Q^2/(2C)e^{-Rt/L}$.

R, L, OR C IN AN AC CIRCUIT

- If we connect a resistor, a capacitor, or an inductor to an external ac (alternating current) emf, $\mathcal{E} = \mathcal{E}_m \sin(\omega_d t)$, ω_d is called the driving frequency. The voltage drop across the element (x = R, C or L) is then $v_x = V_x \sin(\omega_d t)$.
- The current can then be calculated with $i_x = I_x \sin(\omega_d t - \phi)$, where ϕ corresponds to the phase difference between the voltage and the current.
- The equivalence of a resistance for a capacitor and an inductor is the reactance. The capacitive reactance is $X_C = 1/(\omega_d C)$ and the inductive reactance is $X_L = \omega_d L$.

Element	Symbol	Resistance or Reactance	Phase of Current	Phase angle ϕ	Amplitude Relation
Resistor	R	R	in phase	$0°$	$V_R = I_R R$
Capacitor	C	$X_C = 1/(\omega_d C)$	leads v_C (ICE)	$-90°$	$V_C = I_C X_C$
Inductor	L	$X_L = \omega_d L$	lags v_L (ELI)	$90°$	$V_L = I_L X_L$

RLC CIRCUIT

- In an RLC circuit the instantaneous voltages have to add up to the emf: $\mathcal{E} = v_R + v_L + v_C$.
- The amplitudes can be calculated from the vector sum of the phasors. Thus the maximum current is given by $I = \mathcal{E}_m/Z$ where $Z = \sqrt{R^2 + (X_L - X_C)^2}$ is the impedance of the circuit.
- The phase constant is defined as $\tan\phi = (X_L - X_C)/R$.
- The current has a maximum when the impedance has a minimum, i.e. $Z = R$. This occurs at the resonance frequency $\omega_d = \omega = 1/\sqrt{LC}$.
- The average power dissipated in an ac circuit is $P_{av} = I_{rms}^2 R$, where the rms current is $I_{rms} = I/\sqrt{2}$. The definition of an rms emf as $\mathcal{E}_{rms} = \mathcal{E}_m/\sqrt{2}$ yields also: $P_{av} = \mathcal{E}_{rms} I_{rms} R/Z = \mathcal{E}_{rms} I_{rms} \cos\phi$.

TRANSFORMERS

- Transformers consist of two coils (primary and secondary) wound on the same iron core with different number of turns. The same magnetic flux is then in both coils:
  $V_p = -N_p \Delta\phi_B/\Delta t$ and $V_s = -N_s \Delta\phi_B/\Delta t \rightarrow V_s/V_p = N_s/N_p$.
  The currents are then related via: $I_s/I_p = N_p/N_s$.
- The equivalent resistance of the coil in the primary circuit is given by $R_{eq} = (N_p/N_s)^2 R$, where R is the resistance in the circuit of the second coil.

ELECTROMAGNETIC WAVES

DEFINITION OF EM-WAVES

- The electric and magnetic field produced with an LC oscillator connected to an antenna in the z direction can be described with wave equations (at large distances):
$E = E_m \sin(kx - \omega t)$ and $B = B_m \sin(kx - \omega t)$.
- The electric and magnetic fields are always perpendicular to the direction of travel. It is a transverse wave. The electric field is always perpendicular to the magnetic field.
- The cross product $\vec{E} \times \vec{B}$ always gives the direction of the wave. The electric and the magnetic field are in phase and vary with the same frequency.
- The speed of all electromagnetic waves is given by: $c = 1/\sqrt{\mu_0 \epsilon_0}$, $E_m/B_m = c$, $c = 299,792,458 m/s$.
- The rate of energy transport per unit area is called the pointing vector and is given by: $S = \vec{E} \times \vec{B}/\mu_0$, $S = E^2/(c\mu_0)$.
- The intensity of the wave is defined as $I = \bar{S} = E_{rms}^2/(c\mu_0)$, where E_{rms} is the root-mean-square value of the electric field: $E_{rms} = E_m/\sqrt{2}$. The intensity as a function of distance from the source is given by: $I = P_s/(4\pi r^2)$, where P_s is the power emitted by the source.
- The radiation pressure of a wave is defined as $P_r = I/c$ for total absorption and $P_r = 2I/c$ for total reflection.

POLARIZATION

- The electric field component of a wave parallel to the polarizing direction of a polarizer is passed (transmitted) by the polarizer, the component perpendicular to it is absorbed.
- The intensity of an unpolarized wave after a polarizer is reduced by a factor of 2: $I = I_0/2$.
- The intensity of a polarized wave going through a polarizer is given by $I = I_0 \cos^2\theta$, where θ is the angle between the polarization direction of the wave and the polarizer.

REFLECTION AND REFRACTION

- For electromagnetic waves reflected off surfaces the law of reflection is valid: $\theta_i = \theta_r$ where the incident i and reflected r ray (wave) is measured with respect to the normal of the surface.
- For transparent materials the wave is refracted: $n_1 \sin\theta_1 = n_2 \sin\theta_2$, where n is called the index of refraction for a given material, and 1 and 2 correspond to the incoming and refracted ray, respectively.
- When light (em-wave) travels from an optical denser medium into an optical less dens medium all of the light is reflected at the boundary between the two media when the incoming angle is larger than a critical angle θ_c: $\sin\theta_c = n_2/n_1$,$(n_1 > n_2)$.
- Reflected and refracted light is partially polarized. For a certain angle the reflected light is completely polarized (Brewster angle): $\tan\theta_B = n_2/n_1$.

GEOMETRIC OPTICS

MIRRORS

- The object distance p is located in front of the mirror. Plane mirrors form a *virtual* image behind the mirror at a distance i (= image distance). The mirror image is upright: $p = -i$.
- The magnification m is defined as the ratio of the image height h_i over the object height h_o: $|m| = h_i/h_o$ and is also given by $m = -i/p$. The magnification of a plane mirror is one.
- All spherical mirrors have focal points. The focal length is given by $f = r/2$, where r is the radius of curvature of the mirror.
- The mirror equation relates the focal length, object and image distances: $1/f = 1/p + 1/i$.
  Real images are always on the same side of the mirror as the object.

LENSES

- Symmetric spherical lenses have two symmetrically positioned focal points, one on each side. The distance between the center plane of the lens and the focal point is the focal length, f.
- The mirror equation is also valid for thin lenses: $1/f = 1/o + 1/i$,
  and the magnification is also $m = -i/o$.
  Real images are always formed on the other side of the lens as the object.
- For lenses with two different curvatures on both sides the focal length can be calculated with the lens makers equation: $1/f = (n-1)(1/r_1 - 1/r_2)$. When the object faces a convex refracting surface r is positive. When it faces a concave surface it r is negative.
- For a system of several lenses (or mirrors) the total magnification is given by the product of the individual magnifications: $M = m_1 m_2 m_3 ...$

MIRRORS AND SINGLE LENSES

Thin Lenses		diverging	converging			
Location	Object o (all +)	any value	o<f	2f>o>f	o=2f	o>2f
	Image i	$\lvert i\rvert < \lvert f\rvert$	any value	i > 2f	i=2f	2f>i>f
Image type	relative size	smaller	bigger	bigger	equal	smaller
	real/virtual	virtual	virtual	real	real	real
	erect/inverted	erect	erect	inverted	inverted	inverted
Signs	focal length f	−	+	+	+	+
	image position	−	−	+	+	+
	magnification	+	+	−	−	−
Spherical Mirrors		convex	concave			

WAVE OPTICS

WAVEFRONTS AND REFRACTION

- Points of constant phase are called wavefronts. Wavefronts striking a surface reflect according to $\theta_i = \theta_r$. The velocity of a wave is given by the frequency and wavelength: $v = f\lambda$.
- When a wave enters a medium with a different index of refraction n the frequency does not change but the velocity does: $n = c/v$, where c is the speed of light in vacuum (air) and v is the velocity of light inside the medium. The wavelength also changes: $\lambda_n = \lambda/n$

COHERENT LIGHT AND INTERFERENCE

- Coherent light is light emitted with one particular phase. Conventional light is produced by emission from individual atoms with random phases and is therefore incoherent. LASERs are coherent.
- If we split a light beam of a fixed wavelength into two beams, then they will interfere constructively or destructively depending on the pathlength difference ΔL:
  Constructive Interference: $\Delta L = m\lambda$ with $m = 0, 1, 2, \ldots$
  Destructive Interference: $\Delta L = (m + 1/2)\lambda$ with $m = 0, 1, 2, \ldots$

DOUBLE-SLIT INTERFERENCE

- Monochromatic light on a mask with two thin slits which are separated by a distance d will produce an interference pattern:
  Constructive interference (bright fringes): $d \sin\theta = m\lambda$ $m = 0, 1, 2, \ldots$
  Destructive interference (dark fringes): $d \sin\theta = (m + 1/2)\lambda$ $m = 0, 1, 2, \ldots$
- If D is the distance between screen and the slits, then the location of maxima and minima on the screen with respect to the central maximum is given by $y_m = D\tan\theta_m = D\sin\theta_m = mD\lambda/d$. The approximation $\tan\theta = \sin\theta = \theta$ (in radians) is valid for small angles.
- The intensity I on the screen as a function of the angle θ is given by $I = 4I_0\cos^2(\phi/2)$, where I_0 is the intensity of a single slit and $\phi = (2\pi d/\lambda)\sin\theta$.

THIN-FILM INTERFERENCE

- Light incident on a thin film reflects partly from the front surface, and enters partly the glass which is then reflected from the back surface. Since both light beams come from the same source, they have a fixed phase relationship which depends on the difference in path lengths.
- In the medium, the wavelength changes from λ to $\lambda_2 = \lambda/n_2$. In addition, if light is reflected by a surface to a medium with a higher n, its phase changes by 180°, and if light is reflected on a surface to a medium with lower n, the phase remains unchanged.

	$n_1 < n_2 > n_3$ or $n_1 > n_2 < n_3$	$n_1 < n_2 < n_3$ or $n_1 > n_2 > n_3$
Constructive Interference	$2L = (m + 1/2)\lambda/n_2$	$2L = m\lambda/n_2$
Destructive Interference	$2L = m\lambda/n_2$	$2L = (m + 1/2)\lambda/n_2$

SINGLE SLIT DIFFRACTION

- There is no fundamental difference between diffraction and interference.
- One talks of diffraction when there are many waves involved in the interference instead of only two.
- If one illuminate a thin slit with a beam of light one observes in addition to the central bright spot, other (much less) bright spots on the sides, separated by interference minima. This pattern is due to interference between the light-wavelets going through the slit.
- Angles for diffraction minima: $m\lambda = a \sin\theta \;\; m = 1, 2, 3, \ldots$ where a is the slit width.
- Note: $m = 0$ is missing in these formulas – The bright central maximum is located there. The intensities of the noncentral maxima are much less than that of the central maximum. The width of the central maximum is twice that of the noncentral maxima.
- The intensity I on the screen as a function of the angle θ is given by $I = I_m(\sin\alpha/\alpha)^2$, where I_m is the maximum intensity which occurs at the center and $\alpha = \phi/2 = (\pi a/\lambda)\sin\theta$.

DIFFRACTION BY A DOUBLE SLIT

- For a double slit with finite slit widths a the intensity is given by a combination of the single slit diffraction and the double slit interference: $I = I_m(\cos^2\beta)(\sin\alpha/\alpha)^2$, with $\beta = (\pi d/\lambda)\sin\theta$ and $\alpha = (\pi a/\lambda)\sin\theta$, where d is the distance between the two slits and a is the width of the slits.

DIFFRACTION GRATING

- A diffraction grating is a mask containing a very large number of parallel slits at equal distances d.
- Waves from two neighboring slits will be in phase whenever: $\sin\theta = m\lambda/d \quad m = 0, 1, 2\ldots$
- In the two-slit interference, destructive interference would only be given at exactly the midpoint of two consecutive maxima (path difference = $\lambda/2$). For a diffraction grating, there are many more possible conditions of destructive interference. Thus there are sharp maxima at angles given by the above equation, and between them, there is destructive interference.
- The number of slits is by one larger than the number of secondary minima between primary maxima.
- The interference pattern of the slit separation and the individual slit widths add up to the total interference pattern.