Chapter 7

Rotational Motion
Angles, Angular Velocity and Angular Acceleration
Universal Law of Gravitation
Kepler’s Laws
Angular Displacement

- Circular motion about \textit{AXIS}
- Three different measures of angles:
  1. Degrees
  2. Revolutions (1 rev. = 360 deg.)
  3. Radians (2\(\pi\) rad.s = 360 deg.)
Angular Displacement, cont.

- Change in distance of a point:

\[ s = 2\pi r N \quad (N \text{ counts revolutions}) \]
\[ = r \theta \quad (\theta \text{ is in radians}) \]
**Example**

An automobile wheel has a radius of 42 cm. If a car drives 10 km, through what angle has the wheel rotated?

a) In revolutions

b) In radians

c) In degrees
Solution

Note distance car moves = distance outside of wheel moves

a) Find N:

Known: \( s = 10\,000 \text{ m}, r = 0.42 \text{ m} \)

\[
N = \frac{s}{2\pi r} = \frac{10\,000}{2\pi \times 0.42} = 3\,789
\]

b) Find \( \theta \) in radians

Known: \( N \)

\[\theta = 2\pi \text{(radians/revolution)} \times N = 2.38 \times 10^4 \text{ rad.}\]

c) Find \( \theta \) in degrees

Known: \( N \)

\[\theta = 360 \text{(degrees/revolution)} \times N = 1.36 \times 10^6 \text{ deg}\]
Angular Speed

- Can be given in
  - Revolutions/s
  - Radians/s --> Called $\omega$
  - Degrees/s

$$\omega = \frac{\theta_f - \theta_i}{t} \text{ in radians}$$

- Linear Speed at $r$

$$v = 2\pi r \cdot \frac{\theta_f - \theta_i}{t} \text{ (in revolutions)}$$

$$= \frac{2\pi r \cdot \theta_f - \theta_i}{360} \cdot \frac{1}{t} \text{ (in degrees)}$$

$$= \frac{2\pi r \cdot \theta_f - \theta_i}{2\pi} \cdot \frac{1}{t} = \omega r \text{ (in radians)}$$
Example

A race car engine can turn at a maximum rate of 12 000 rpm. (revolutions per minute).

a) What is the angular velocity in radians per second.

b) If helicopter blades were attached to the crankshaft while it turns with this angular velocity, what is the maximum radius of a blade such that the speed of the blade tips stays below the speed of sound.
DATA: The speed of sound is 343 m/s
Solution

a) Convert rpm to radians per second

\[
12000 \left( \frac{\text{rev.}}{\text{min}} \right) \cdot \frac{2\pi}{60 \left( \frac{\text{sec}}{\text{min}} \right)} \left( \frac{\text{rad}}{\text{rev}} \right) = 1256 \text{ radians/s}
\]

b) Known: \( v = 343 \text{ m/s}, \ \omega = 1256 \text{ rad./s} \)

Find \( r \)

Basic formula

\( v = \omega r \)

\[
r = \frac{v}{\omega} = .27 \text{ m}
\]
Angular Acceleration

- Denoted by $\alpha$

$$\alpha = \frac{\omega_f - \omega_i}{t}$$

- $\omega$ must be in radians per sec.
- Units of angular acceleration are rad/s$^2$
- Every portion of the object has same angular speed and same angular acceleration
## Analogies Between Linear and Rotational Motion

<table>
<thead>
<tr>
<th>Rotational Motion</th>
<th>Linear Motion</th>
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<tbody>
<tr>
<td>$\Delta \theta = \frac{(\omega_i + \omega_f)}{2} t$</td>
<td>$\Delta x = \frac{(v_i + v_f)}{2} t$</td>
</tr>
<tr>
<td>$\Delta \theta = \omega_i t + \frac{1}{2} \alpha t^2$</td>
<td>$\Delta x = v_i t + \frac{1}{2} at^2$</td>
</tr>
<tr>
<td>$\omega = \omega_i + \alpha t$</td>
<td>$v = v_i + at$</td>
</tr>
<tr>
<td>$\omega^2 = \omega_i^2 + 2\alpha \Delta \theta$</td>
<td>$v^2 = v_i^2 + 2a \Delta x$</td>
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</table>
Linear movement of a rotating point

- Distance
  \[ s = \theta r \]
- Speed
  \[ v = \omega r \]
- Acceleration
  \[ a = \alpha r \]

Different points on the same object have different linear motions!

Only works when \( \theta, \omega \) and \( \alpha \) are in radians!
Example

A pottery wheel is accelerated uniformly from rest to a rate of 10 rpm in 30 seconds.

a.) What was the angular acceleration? (in rad/s$^2$)

b.) How many revolutions did the wheel undergo during that time?
Solution

First, find the final angular velocity in radians/s.

\[ \omega_f = 10 \left( \frac{\text{rev.}}{\text{min}} \right) \cdot \frac{1}{60(\text{sec/ min})} \cdot 2\pi \left( \frac{\text{rad}}{\text{rev}} \right) = 1.047 \left( \frac{\text{rad}}{\text{sec}} \right) \]

a) Find angular acceleration

Basic formula

\[ \omega_f = \omega_i + \alpha t \]

\[ \alpha = \frac{\omega_f - \omega_i}{t} = 0.0349 \text{ rad./s}^2 \]

b) Find number of revolutions: Known \( \omega_i=0, \omega_f=1.047, \text{ and } t = 30 \)

First find \( \Delta \theta \) in radians

Basic formula

\[ \Delta \theta = \frac{\omega_i + \omega_f}{2} t = 15.7 \text{ rad.} \]

\[ N = \frac{\Delta \theta(\text{rad.})}{2\pi(\text{rad./rev.})} = 2.5 \text{ rev.} \]
b) Find number of revolutions:

Known $\omega_i = 0$, $\omega_f = 1.047$, and $t = 30$, 

First find $\Delta \theta$ in radians 

$$\Delta \theta = \frac{\omega_f}{2} t = 15.7 \text{ rad.}$$

$$N = \frac{\Delta \theta \text{(rad.)}}{2\pi \text{(rad./rev.)}} = 2.5 \text{ rev.}$$
A coin of radius 1.5 cm is initially rolling with a rotational speed of 3.0 radians per second, and comes to a rest after experiencing a slowing down of $\alpha = 0.05 \text{ rad/s}^2$.

a.) Over what angle (in radians) did the coin rotate?

b.) What linear distance did the coin move?
Solution

a) Find $\Delta \theta$, Given $\omega_i = 3.0 \text{ rad/s}, \omega_f = 0, \alpha = -0.05 \text{ rad/s}^2$

Basic formula

$$\omega_f^2 = \omega_i^2 + 2\alpha \Delta \theta$$

$$\Delta \theta = \frac{\omega_i^2}{-2\alpha}$$

= 90 radians = $90/2\pi$ revolutions

b) Find $s$, the distance the coin rolled

Given: $r = 1.5 \text{ cm}$ and $\Delta \theta = 90 \text{ rad}$

Basic formula

$$s = r\Delta \theta$$

$s = r\Delta \theta$, $(\Delta \theta \text{ is in rad.s}) = 135 \text{ cm}$
Centripetal Acceleration

- Moving in circle at constant *SPEED* does not mean constant *VELOCITY*
- Centripetal acceleration results from *CHANGING DIRECTION* of the velocity
Centripetal Acceleration, cont.

- Acceleration is directed toward the center of the circle of motion

Basic formula
\[ \vec{a} = \frac{\Delta \vec{v}}{t} \]
Derivation: \( a = \omega^2 r = \frac{v^2}{r} \)

From the geometry of the Figure

\[ \Delta v = 2v \sin(\Delta \theta / 2) \]
\[ = v \Delta \theta \quad \text{for small } \Delta \theta \]

From the definition of angular velocity

\[ \Delta \theta = \omega t \]
\[ \Delta v = v \omega t \]

\[ a = \frac{\Delta v}{t} = v \omega = \omega^2 r = \frac{v^2}{r} \]
Forces Causing Centripetal Acceleration

- Newton’s Second Law
  \[ \vec{F} = m\vec{a} \]
- Radial acceleration requires radial force
- Examples of forces
  - Spinning ball on a string
  - Gravity
  - Electric forces, e.g. atoms
Example

A space-station is constructed like a barbell with two 1000-kg compartments separated by 50 meters that spin in a circle (r=25 m). The compartments spins once every 10 seconds.

a) What is the acceleration at the extreme end of the compartment? Give answer in terms of “g”s.

b) If the two compartments are held together by a cable, what is the tension in the cable?
Solution

a) Find acceleration $a$

Given: $T = 10 \text{ s}$, $r = 25 \text{ m}$

First, find $\omega$ in rad/s

$$\omega = 2\pi \cdot 0.1$$

Then, find acceleration $a$

$$a = \omega^2 r = 9.87 \text{ m/s}^2 = 1.006 \text{ g}$$
Solution

b) Find the tension

Given $m = 1000 \text{ kg}$, $a = 1.006 \text{ g}$

Basic formula

$$F = ma$$

$$T = ma = 9870 \text{ N}$$
Example

A race car speeds around a circular track.

a) If the coefficient of friction with the tires is 1.1, what is the maximum centripetal acceleration (in “g”s) that the race car can experience?

b) What is the minimum circumference of the track that would permit the race car to travel at 300 km/hr?
Solution

a) Find the maximum centripetal acceleration

Known: $\mu = 1.1$

Remember, only consider forces towards center

Basic formula

\[ f = \mu n \]
\[ F = ma \]
\[ ma = \mu mg \]
\[ a = \mu g \]

Maximum $a = 1.1 \, g$
Solution

b) Find the minimum circumference

Known: \( v = 300 \text{ km/hr} = 83.33 \text{ m/s} \), \( a = 1.1 \text{ g} \)

First, find radius

Basic formula

\[
a = \frac{v^2}{r} = \omega^2 r
\]

Then, find circumference

\[
L = 2\pi r = 4043 \text{ m}
\]

In the real world: tracks are banked
Example

A yo-yo is spun in a circle as shown. If the length of the string is \( L = 35 \) cm and the circular path is repeated 1.5 times per second, at what angle \( \theta \) (with respect to the vertical) does the string bend?
Solution

Apply \( F = ma \) for both the horizontal and vertical components.

Basic formula
\[
\vec{F} = m\vec{a}
\]

Apply \( F = ma \) for both the horizontal and vertical components.

Basic formula
\[
a = \omega^2 r
\]

\[
\begin{align*}
ma_y &= 0 \\
\sum F_y &= T \cos \theta - mg \\
T \cos \theta &= mg
\end{align*}
\]

\[
\begin{align*}
ma_x &= m\omega^2 r = m\omega^2 L \sin \theta \\
\sum F_x &= T \sin \theta \\
m\omega^2 L &= T
\end{align*}
\]

\[
r = L \sin \theta
\]
Solution

We want to find $\theta$, given $\omega = 2\pi \cdot 1.5$ & $L = 0.35$

\[ ma_x = m\omega^2 r = m\omega^2 L \sin \theta \]
\[ \sum F_x = T \sin \theta \]
\[ m\omega^2 L = T \]

\[ ma_y = 0 \]
\[ \sum F_y = T \cos \theta - mg \]
\[ T \cos \theta = mg \]

2 eq.s & 2 unknowns (T and $\theta$)

\[ \cos \theta = \frac{mg}{T} = \frac{g}{\omega^2 L} \]

$\theta = 71$ degrees
Accelerating Reference Frames

Consider a frame that is accelerating with $a_f$

\[ F = ma \]

\[ F - ma_f = m(a - a_f) \]

Fictitious force
Looks like "gravitational" force

If frame acceleration = $g$,
fictitious force cancels real gravity.

Examples: Falling elevator, planetary orbit
rotating space stations
DEMO: FLYING POKER CHIPS
Which of these astronauts experiences “zero gravity”?

a) An astronaut billions of light years from any planet.

b) An astronaut falling freely in a broken elevator.

c) An astronaut orbiting the Earth in a low orbit.

d) An astronaut far from any significant stellar object in a rapidly rotating space station.
Newton’s Law of Universal Gravitation

- Force is always attractive
- Force is proportional to both masses
- Force is inversely proportional to separation squared

\[ F = G \frac{m_1 m_2}{r^2} \]

\[ G = 6.67 \times 10^{-11} \left( \frac{m^3}{kg \cdot s^2} \right) \]
Gravitation Constant

- Determined experimentally
- Henry Cavendish, 1798
- Light beam / mirror amplify motion
Example

Given: In SI units, \( G = 6.67 \times 10^{-11} \), \( g = 9.81 \) and the radius of Earth is \( 6.38 \times 10^6 \).
Find Earth’s mass:

\[
mg = G \frac{Mm}{R^2} = mg
\]

\[
M = \frac{gR^2}{G} = 5.99 \times 10^{24} \text{ kg}
\]
Example

Given: The mass of Jupiter is $1.73 \times 10^{27}$ kg and Period of Io’s orbit is 17 days
Find: Radius of Io’s orbit
Solution

Given: \( T = 17 \cdot 24 \cdot 3600 = 1.47 \times 10^6 \), \( M = 1.73 \times 10^{27} \), \( G = 6.67 \times 10^{-11} \)

Find: \( r \)

First, find \( \omega \) from the period

\[
\omega = 2\pi \frac{1}{T} = 4.28 \times 10^{-6} \text{ s}
\]

Next, solve for \( r \)

\[
\omega^2 r = G \frac{M}{r^2}
\]

\[
r^3 = \left( \frac{GM}{\omega^2} \right)
\]

\[
r = 1.84 \times 10^9 \text{ m}
\]
Tycho Brahe (1546-1601)

- Lost part of nose in a duel
- EXTREMELY ACCURATE astronomical observations, nearly 10X improvement, corrected for atmosphere
- Believed in *Retrograde Motion*
- Hired Kepler to work as mathematician
Johannes Kepler (1571-1630)

- First to:
  - Explain planetary motion
  - Investigate the formation of pictures with a pin hole camera;
  - Explain the process of vision by refraction within the eye
  - Formulate eyeglass designed for nearsightedness and farsightedness;
  - Explain the use of both eyes for depth perception.
  - First to describe: real, virtual, upright and inverted images and magnification
First to:

- explain the principles of how a telescope works
- discover and describe total internal reflection.
- explain that tides are caused by the Moon.
- He tried to use stellar parallax caused by the Earth's orbit to measure the distance to the stars; the same principle as depth perception. Today this branch of research is called astrometry.
- suggest that the Sun rotates about its axis
- derive the birth year of Christ, that is now universally accepted.
- derive logarithms purely based on mathematics,
Isaac Newton (1642-1727)

- Invented Calculus
- Formulated the universal law of gravitation
- Showed how Kepler’s laws could be derived from an inverse-square-law force
- Invented Wave Mechanics
- Numerous advances to mathematics and geometry
Kepler’s Laws

1. All planets move in elliptical orbits with the Sun at one of the focal points.

2. A line drawn from the Sun to any planet sweeps out equal areas in equal time intervals.

3. The square of the orbital period of any planet is proportional to cube of the average distance from the Sun to the planet.
Kepler’s First Law

- All planets move in elliptical orbits with the Sun at one focus.
  - Any object bound to another by an inverse square law will move in an elliptical path
- Second focus is empty
Kepler’s Second Law

- A line drawn from the Sun to any planet will sweep out equal areas in equal times
  - Area from A to B and C to D are the same

This is true for any central force due to angular momentum conservation (next chapter)
Kepler’s Third Law

- The square of the orbital period of any planet is proportional to cube of the average distance from the Sun to the planet.

\[ \frac{T^2}{r^3} = K_{\text{sun}} \]

- For orbit around the Sun, \( K_S = 2.97 \times 10^{-19} \) s\(^2\)/m\(^3\)

- \( K \) is independent of the mass of the planet
Derivation of Kepler’s Third Law

Basic formula
\[ F = ma = G \frac{Mm}{r^2} \]
\[ a = \omega^2 r \]

\[ m\omega^2 r = G \frac{Mm}{r^2} \]

Basic formula
\[ \omega = \frac{2\pi}{T} \]

\[ r^3 = \frac{GM}{4\pi^2} T^2 \]
\[ \frac{T^2}{r^3} = \frac{4\pi^2}{GM} = K_{\text{sun}} \]
Example

Data: Radius of Earth’s orbit = 1.0 A.U.
    Period of Jupiter’s orbit = 11.9 years
    Period of Earth’s orbit = 1.0 years
Find: Radius of Jupiter’s orbit

Basic formula

\[ \frac{T_{Earth}^2}{r_{Earth}^3} = \frac{T_{Jupiter}^2}{r_{Jupiter}^3} \]

\[ r_{Jupiter}^3 = r_{Earth}^3 \left( \frac{T_{Jupiter}}{T_{Earth}} \right)^{2/3} \]

\[ r_{Jupiter} = r_{Earth} \left( \frac{T_{Jupiter}}{T_{Earth}} \right)^{2/3} = 5.2 \text{ A.U.} \]
Gravitational Potential Energy

- PE = mgy is valid only near the Earth’s surface.
- For arbitrary altitude,

\[ PE = -G \frac{M_E m}{r} \]

- Zero reference level is infinitely far from the earth.
Graphing PE vs. position

\[ PE = -G \frac{M_E m}{r} \]
You wish to hurl a projectile is hurled from the surface of the Earth \((R_e = 6.3 \times 10^6 \text{ m})\) to an altitude of \(20 \times 10^6 \text{ m}\) above the surface of the Earth. Ignore the rotation of the Earth and air resistance.

a) What initial velocity is required?

b) What velocity would be required in order for the projectile to reach infinitely high? I.e., what is the escape velocity?
Solution

Given: \( R_0 = 6.3 \times 10^6 \), \( R = 26.3 \times 10^6 \), \( G, M = 6.0 \times 10^{24} \)

Find: \( v_0 \)

First, get expression for change in PE

\[
\Delta PE = G M m \left( \frac{1}{r_0} - \frac{1}{r} \right)
\]

Then, apply energy conservation

\[
\Delta PE = \Delta KE
\]

\[
G M m \left( \frac{1}{r_0} - \frac{1}{r} \right) = \frac{1}{2} m v_0^2
\]

Finally, solve for \( v_0 = 6600 \text{ m/s} \)
**Solution For Escape Velocity**

Given: \( R_0 = 6.3 \times 10^6, \ R = \infty, \ G, \ M = 6.0 \times 10^{24} \)

Find: \( v_0 \)

First, get expression for change in PE

Basic formula

\[
PE = -G \frac{Mm}{r}
\]

\[
\Delta PE = GMm \frac{1}{r_0}
\]

Then, apply energy conservation

Basic formula

\[
KE = \frac{1}{2} mv^2
\]

\[
GMm \frac{1}{r_0} = \frac{1}{2} mv_0^2
\]

Solve for \( v_0 = 11 \ 270 \text{ m/s} \)