Chapter 2: Spring forces

A. The properties of an ideal spring

In this chapter, a language and a notation to describe all forces is developed from the behavior of elastic forces. The relationship governing elastic objects and their forces was first codified by Robert Hooke, and is therefore called *Hooke's law*:

> An elastic object will distort (stretch or compress) by an amount that is proportional to applied force.

Most solid objects behave elastically for small applied forces. Elastic objects return quickly to their normal shape when distorting forces are removed. Distorting forces that are too large, however, can cause a permanent change in shape. Objects that do not return to their natural shape have been distorted beyond their "elastic limit".

Before the details of how to apply Hooke's law are presented, the general features of the forces applied to, and generated by a spring will be investigated. For now, consider only objects that can be represented by an "ideal spring", a light (i.e., massless) but strong coil spring; a spring from a retractable ballpoint pen, or from a mattress are accessible examples. Elastic objects with significant mass retain many of these properties; they are considered later in this text to simplify the discussion.

What is not stated in Hooke's law, is that the distorting force(s) must be applied to both ends of the ideal spring! Remove a spring from a ballpoint pen and hold it at only one end, with the other end free (not held or attached to another object). See how far you can stretch it. No matter how hard you *attempt* to pull or push on the spring, the spring cannot be distorted by any significant amount. Pushing with an open hand on one end,

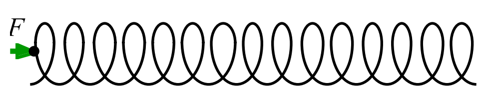


Figure 2.1 An attempt to apply different forces to the ends of an ideal spring, will fail.

leaving the other end free causes the spring to leave your hand rapidly without compressing, as shown in Figure 2.1. On the other hand, if you put two opposing forces of equal strength on opposite ends of the spring you can maintain the spring fixed in front of you, with a visible change in length. As demonstrated by this experiment and explored further in the next section, forces applied to ideal springs must conform to three rules:

> a) Forces *cannot* be applied to only one end of an ideal spring,
> b) An ideal spring will distort if, and only if forces are applied to both ends, and
> c) Forces applied to an ideal spring must be opposing forces of equal strength.

B. Action and Reaction Forces

In the following discussion consider only springs that are strong, light (massless), and elastic. A "slinky", looks like a spring, but it is not elastic. After distorting forces are removed, it does not return to its original shape. From the experimental evidence of the previous chapter, it appears that two opposing forces of equal strength, one acting at each end, are needed to maintain an ideal spring in either a stretched or compressed state, as is

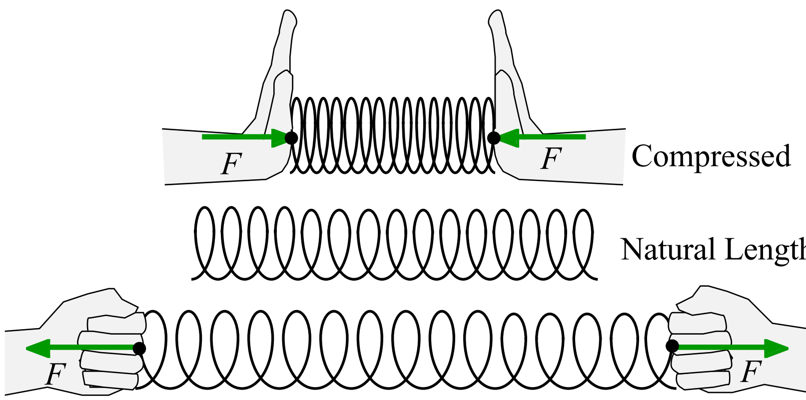


Figure 2.2 A spring in compressed, natural, and stretched states, and the forces, as applied by the hands of a person, necessary to maintain those states.

shown in Figure 2.2.

The compressed state (top) is obtained by pushing in on both ends of the spring, a natural state (middle) with no forces acting on the spring, and a stretched or elongated state (bottom), obtained by pulling on the ends of the spring. The forces generated by a person's hand are represented as an arrow "attached" to the spring at a point marked by the black dot. Allowing some artistic license, the hands are removed in Figure 2.3, to show

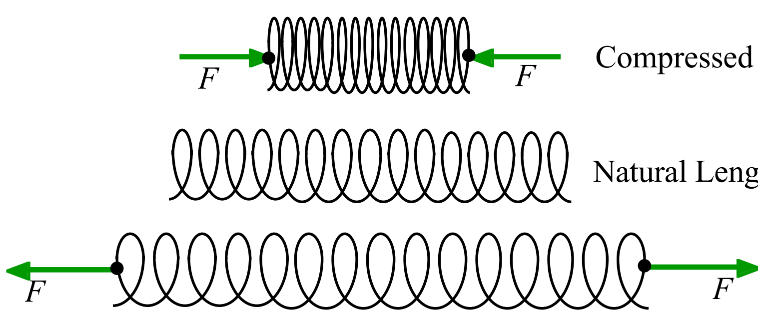


Figure 2.3 A spring held by forces (vectors with magnitude, F) in compressed, natural ($F = 0$), and stretched states.

only forces acting on the springs. Any source applying these forces to the spring will distort it in the same way. In contrast, the object on which the forces act cannot be removed; *a force must always act on an object*. You must push or pull *on something*!

The arrows are called force *vectors*. The arrow *points* in the direction that the force acts, with the head or tail touching the object where the force is applied. The strength of a force vector is its *magnitude*, specified here in two ways: a) with an algebraic symbol, the italic letter, F, having a precise, *positive* value equal to the strength of the force, and b) with an arrow drawn with a length proportional to the strength of the force. The vector arrows are not drawn with great precision, but the relative size, e.g., large, medium, or small, can be useful as a descriptive tool.

The two force vectors acting on a spring in Figure 2.3 have the same magnitude, F (the same length), but act in different directions, one to left and one to right. The two force vectors, therefore, are <u>not</u> equal to each other; the force *magnitudes* are the same, but the force *directions* are different.

Two force vectors are equal if, and only if, they have the same magnitude *and* direction.

The forces, shown in Figure 2.3, *pull* on the spring from outside its ends. If the ends are *pushed* outward, i.e., away from the center of the spring the spring will also stretch. Never-the-less, a convention will be used, where a force acting *on* a spring, has a vector arrow drawn *outside* the spring, while a force generated *by* a spring, has a vector arrow drawn *within* it, and acting *on* the object at the spring end. Later, to display large forces generated by small objects, this convention may require a magnified scale.

In response (or reaction) to being stretched or compressed by the applied forces, the spring generates the internal forces of compression (C), and tension (T) shown in Figure 2.4. A compressed (squeezed) spring generates compression forces, vectors shown

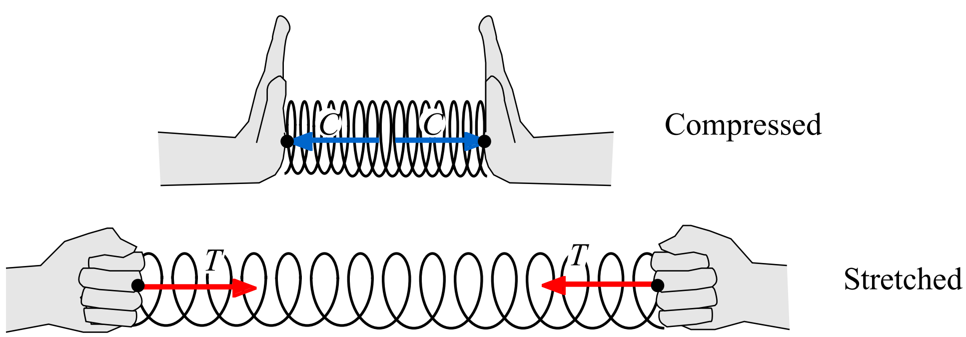


Figure 2.4 A compressed spring generates compression forces (C) that push the hands outward. A stretched spring generates tension forces (T) that pull the hands inward.

labeled C, that push (outward) on the objects that maintain the compression spring. A stretched spring generates tension forces, vectors shown labeled T, that pull (inward) on the objects that maintain the stretch of the spring. The italic letters, F, C, or T *label* the force vector (as in "... the tension, T, acts on...") and have *values* equal to the magnitude of the vector (as in "... the magnitude, T, is equal to...").

The springs can be removed, as shown in Figure 2.5, to exhibit more clearly the reaction forces, acting at a point (the black dot) on the hands. Note that each force acts on a *different* hand!

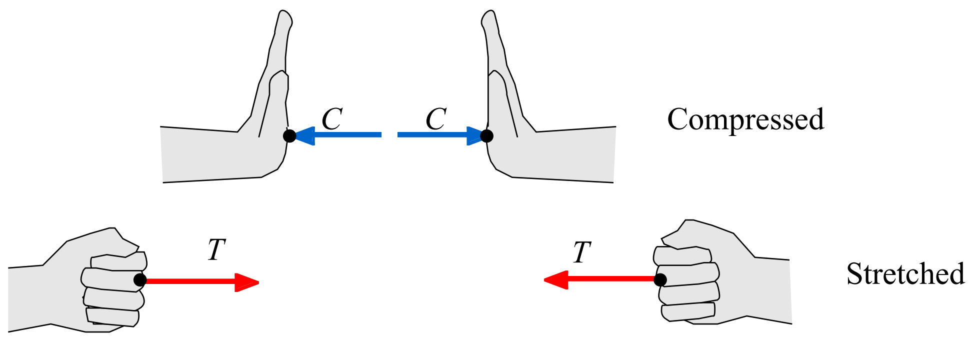


Figure 2.5 A compressed spring generates compression forces (C) that push outward. A stretched spring generates tension forces (T) that pull inward.

An ideal spring, shown in Figure 2.6, reacts by stretching or compressing in response to external forces applied in opposing directions at the ends of the spring. At the same time, the spring generates internal "reaction" forces of tension or compression acting on the hands. The reaction forces produced by the spring are tension forces, T, in the stretched spring and compression forces, C, in the compressed spring

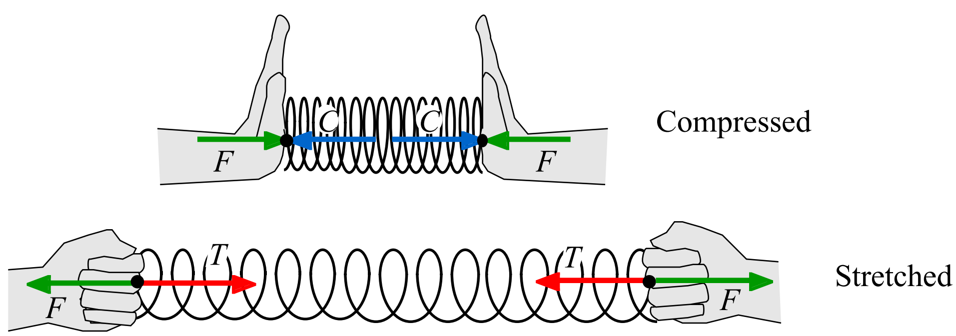


Figure 2.6 The action and reaction forces of a stretched and compressed spring.

The statements above apply to ideal springs in the stationary states shown. Transitions from the natural length to the states shown in Fig. 2.6 are assumed to occur very slowly. The force that the person uses to move the hands has the same magnitude as the force the hands apply to the spring. A natural length ideal spring will instantly stretch by a small amount if the hands are moved a small distance away from the center. To make that change, the person must make both hands apply a small force, F, to the ends of the spring. The change in the spring's length generates small tension forces, T, with an equal magnitude but opposite direction to the applied force.

For a stretched ideal spring the force *magnitudes*, F and T, will be equal ($T = F$) and, for a compressed spring the force *magnitudes*, F and C, will be equal ($C = F$).

The stretched spring somehow seems to know just how far to let the hands separate and just how much reactive force is needed to stop the distortion. A spring can't "know" anything, and yet it appears as if the spring makes a decision to generate just the right amount of force. This illusion is caused by a feedback mechanism involving the motion of the hands. Any tendency for the hands to move to a slightly larger separation, by slightly increasing the force applied, results in a corresponding increase in reaction forces that resist that motion. You can try to imagine a very short time when the forces applied by the person to the hands and the reaction forces of the spring do not have the same magnitude. At the end of this time, you will find the force magnitudes are again equalized. In fact the two forces have the same magnitude at *all* times.

If the stretching is rapid, the hands (with mass) attached at the end of the spring cannot not stop instantly and will continue past the point determined by the force the person applies on the hands. For now, consider massless springs and motion sufficiently slow, that the hands move very slowly and can stop rapidly.

C. Newton's 3rd law

The mirror-like behavior of forces acting at the ends of a spring were generalized by Newton in his 3rd law governing the forces at a point of contact between two objects:

Between *two* objects (#1 and #2), an action force (*by* #1, *on* #2) is paired with a reaction force (*by* #2, *on* #1) of equal magnitude and opposite direction.

For our ideal spring, it is natural to associate an "action" force with one of the external forces *acting on* the spring, while the "reaction" is the internal tension or compression force *generated by* the spring at the same end. The internal and external forces occur simultaneously, and either can be considered an action or reaction force.

At a *contact* point between objects, an action - reaction pair of forces always exists.

In these examples, this occurred whenever a hand touched the end of a spring. The forces applied by the two hands should not be considered an action - reaction pair because they both act on the spring.

An action - reaction pair of forces must act on *different* objects.

The 3rd Law is true in all states of motion of two objects in contact: 1) at rest (stationary, not moving) 2) moving with constant speed and direction, or 3) accelerating (speed or direction changing). At a point where a person pulls or pushes on an object, forces that obey Newton's 3rd law are generated through distortion of the materials at the interface. The response of a slinky or chocolate pudding to a distorting force can be very difficult to analyze, however, at the contact point of the distorting force, a reaction force exists with the magnitude and direction required by Newton's 3rd law.

To become familiar with elastic forces and Newton's 3rd laws of forces, consider an example that brings together the effects of compression and tension to create a stable combination. In the model described in Chapter 1, all solid materials act like a spring under tension and compression as long as these forces are not too large. A stiff rod, shown stationary and being compressed by two hands in Figure 2.7, is a very strong spring. The rod will react to the applied forces, magnitude *F*, by generating compression

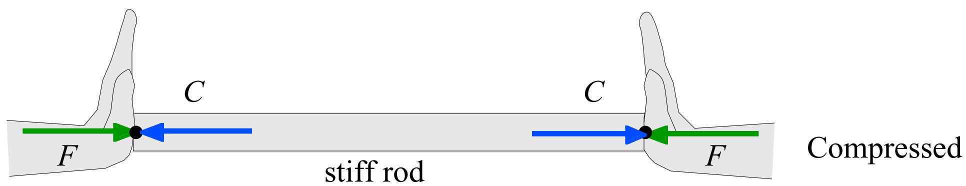


Figure 2.7 A stiff rod compressed by two hands is generating compression forces, magnitude *C*, that act on the hands.

forces, magnitude *C*, that act on the hands and resist further compression of the rod. Compressed by a distance so small that you'd need a microscope to see it, the rod can generate these large compression forces.

A strong spring is shown in Figure 2.8, stretched by outward forces of the same

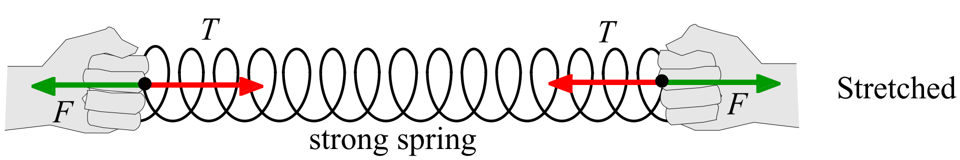


Figure 2.8 A strong spring stretched by two hands is generating tension forces, magnitude *T*, that act on the hands.

magnitude, *F*. The spring generates tension forces, magnitude *T*, that act back on the hands resisting further stretching.

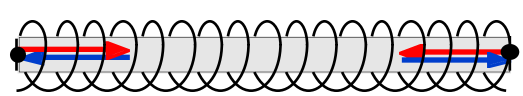


Figure 2.9. The internal forces acting in a stretched spring and a compressed rod attached at their ends.

Distorted by the same magnitude of force, *F*, the compressed rod of Figure 2.7 and stretched spring of Figure 2.8, are now attached, with the rod running through the center of the spring, as shown in Figure 2.9. The spring remains stretched and the rod compressed; the hands are no longer needed. The tension forces of the spring pull inward on the rod to keep it compressed, while the compression forces of the rod push outward on the spring to keep it stretched. An action - reaction pair, consisting of one tension and one compression force, exists at each end of the combination.

The action and reaction forces are indistinguishable and cannot exist without each other; the action force on one, is the reaction force of the other. The elastic forces acting on each object (drawn outside the objects to conform to the previous convention) are shown separately for clarity in Figure 2.10.

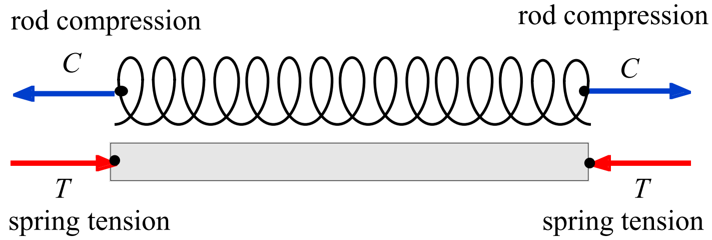


Figure 2.10. Rod compression forces keep the spring stretched while the spring tension keeps the rod compressed.

The construction of a stringed musical instrument, shown in Figure 2.11, is closely related to the rod and spring above. All stringed instruments operate on the same principle: a stretched string vibrates at a characteristic frequency determined by the length

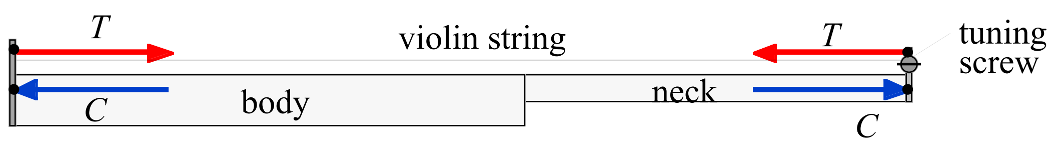


Figure 2.11 Schematic drawing of a stringed musical instrument's parts (violin).

and tension of the string. The string lies directly on the neck and body, though to vibrate freely there must be a small space (created by a bridge). The instrument must bend slightly to counteract the offset of the string from the centerline of the instrument, but that will be ignored in this discussion.

The string tension, adjusted by a screw on which the string has been wound, controls the pitch. The string behaves much like a spring, except its length remains constant as the tension changes. The stretched string pulls inward at both ends with a tension force *T*. A small reduction in length of the neck and body generates large compression forces, labeled *C* in Figure 2.11, that maintain the string tension.

D. Scalars, Vectors and Newton's 1st Law

Vectors, quantities having magnitude and direction, are unavoidable in a description of the physical world. On the other hand, many physical quantities are scalars, fully specified by a single number, perhaps with units. The magnitude of a vector is a scalar with a positive value. Stated emphatically*: the magnitude of a vector cannot be negative.* Thus, the quantity, –10 lb., cannot be the magnitude of a force vector.

Some scalars can have negative values. Examples are temperature, *Temp* = –10 °F (units are degrees Fahrenheit), sound intensity, $I = -10$ dB (the units are decibels), and an altitude, $A = -45$ ft (the units are feet). Some arbitrary choice for zero, e.g., sea level for altitude, determined the sign of these quantities. Many physical quantities, such as work and potential energy, are scalars that can have either negative or positive values, others, such as length and kinetic energy, are positive scalars and cannot take a negative value.

Along a straight line (one dimension) there exist two opposing directions. Examples are 1) "up" and "down", or 2) "left" and "right", etc. In one dimension, the direction of a vector is specified by attaching a sign (+ or –) to the magnitude. In calculations, if you see a negative sign attached to a force vector's magnitude (e.g., $-F$), the direction of the force vector is negative. It's direction points opposite to and 180° away from a force vector, $+F$, that points in the positive direction. The magnitude of a vector must be a positive scalar; changing only its magnitude can never change the direction of a vector.

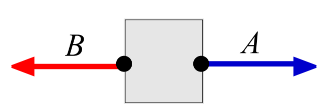
Because all vector magnitudes must be positive, a force of –10 lb., must be the full vector quantity, with a 10 lb. magnitude, and a negative direction. A positive quantity, like 10 lb. (+10 lb.), can be a force vector in the positive direction, or just its magnitude.

A symbol in bold face type, such as **F,** is used to represent a vector quantity. It contains two separate parts, a magnitude and a direction. There are no ambiguities in the statement, **F** = 10 lb.; the magnitude (F) is 10 lb., and the direction is positive. Specifying the force vector as, **F** = –10 lb., has the same magnitude, 10 lb., but a direction that is negative. In handwriting script, on a lecture transparency or a *homework assignment*, a force magnitude is written with a letter, i.e., F. The full vector quantity is written as a letter with an arrow above, $\vec{\mathrm{F}}$ (bold and italic letters being too difficult to consistently reproduce by hand).

In this notation, we can describe a situation where Mr. A pulls to the right with the force, $\mathbf{A} = +A$, and Ms. B pulls to the left with a force, $\mathbf{B} = -B$, where "to the right" is assigned the positive direction (+) and "to the left", the negative (–) direction.
Forces, **A** and **B**, acting on separate boxes can be shown as,

,

and if the forces act on the same box, shown as,



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Acting on the same object, the two force vectors, **A** and **B**, are said to "balance", if the magnitudes, A and B, are equal. In the algebra of vectors, "balance" means that the sum of the two vectors is equal to zero: $\mathbf{A} + \mathbf{B} = 0$. It is intuitively obvious that for this equation to hold the magnitudes of the two forces must be the same and the force directions must be opposite. Also, it can be shown to be true algebraically:

$\mathbf{A} + \mathbf{B} = 0$ Step 1) replace vector symbols with a magnitude and a direction.

$(+A) + (-B) = 0$ Step 2) add vector $(+B)$ to both sides, noting that $(-B) + (+B) = 0$.

$+A = +B$ Therefore, the magnitudes must also be equal, $A = B$.

Two force vectors balance (sum to zero), if: 1) they act on the same object, 2) point in opposite directions and, 3) have equal magnitudes. Each condition must be satisfied for two forces to balance. If the motion of mass is unchanging (stationary, or moving with constant speed and direction) then balanced forces act on it. Newton's laws of forces describe the effects of balanced or unbalanced forces acting on a mass:

1) Motion of a mass remains *unchanged* only by the action of *balanced forces.*
2) Motion (speed or direction) of a mass *changes* by the action of *unbalanced forces.*

Unbalanced forces and applications of Newton's 2nd law are addressed in a later chapter.

E. Relationship between balanced forces and action - reaction pairs.

The tension and compression forces acting where the spring touches the rod are members of an action - reaction pair; each force of the pair acts on a different object; one on the rod and one on the spring. Balanced forces, on the other hand, must act on the same object. If two forces do not act on the *same* object they cannot balance.

To investigate the relationship between balanced forces and action-reaction pairs, the rod and spring, as shown in Figure 2.10, are modified by attaching *massless* hooks to the ends of the rod, and attaching the stretched spring to the hooks, as shown in Figure 2.12. The tension forces generated by the spring pull the hooks inward while the compression forces of the rod push these same hooks outward. In the figure, the forces are shown displaced to either side of the combined spring and rod for clarity, though in reality the forces of tension and compression act through the center of each object, as in Figure 2.9.

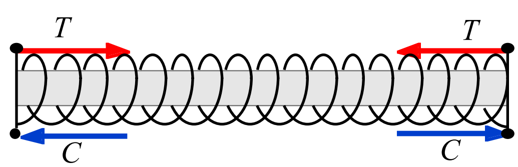

Figure 2.12 Tension forces of the spring and the compression forces of a rod act on massless hooks at the end of the rod. Forces are shown displaced to the side for clarity.

On each hook, the tension and compression forces have equal magnitudes $(T = C)$ and act (on a single object) in opposite directions, and are, therefore, balanced. Previously a tension and compression force comprised an action - reaction pair, but now they are balanced forces acting on massless hooks. By introducing massless hooks (or glue) between the two objects in contact, this relationship between balanced forces and action - reaction pairs can always be found. The forces acting on a massless object (spring, hook, string or wire, if considered massless) will always balance.

Balanced forces acting on an object cause it to compress or stretch. The forces acting on a mass will never *cancel*. For the spring and rod, as shown in Figure 2.12, the forces acting clearly do not "cancel"; they have a considerable effect on the components.

F. The balancing of more than two forces.

The are a limited number of situations where just two forces are in balance. More than two forces acting on a single object are in balance, if the sum of those forces, called the *net force*, is equal to zero:

$$\mathbf{F}_{\text{net}} = \mathbf{F}_1 + \mathbf{F}_2 + \ldots + \mathbf{F}_n = 0 \qquad \text{Eq. 2.1}$$

The next example a spring is stretched and attached on the mid-line between the sides of a *stationary* rectangular frame as shown in Figure 2.13. The pieces of the frame are *not connected* to each other. Only the tension of the spring keeps the two ends of the frame in contact with the upper and lower pieces of the frame. The short ends of the frame are very strong and will be pulled inward by the spring tension, T. The long sides react by generating compression forces, C, that also act on the frame ends. Since all parts of the frame are stationary (or moving with constant speed and direction) the sum of the forces acting on *each end* of the frame must *balance*.

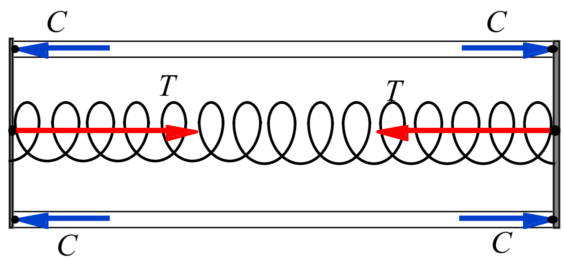

Figure 2.13 Spring stretched in a frame with stiff ends.

The two tension forces in the spring *do not balance*! The two tension forces are equal in magnitude and opposite in direction but they do not act on the same piece of the frame. Forces must act on the same object in order to balance! Focus your attention on only *one end* of the frame at a time (the forces are just reversed on the other end). At each end of the *stationary* frame, by Newton's 1st law, the two compression forces acting in one direction and the single tension force acting in the opposite direction, as shown in Figure 2.14, must balance. The net effect of the *two* compression forces, C, acting

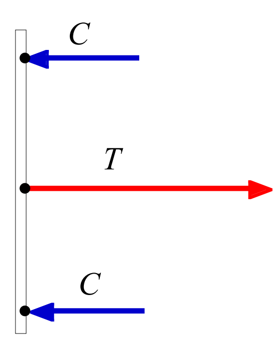

Figure 2.14 Forces acting on the left end of the frame

on one end of the frame must balance the effect of the spring tension, T, also acting on that end. To accomplish this balancing the magnitudes of the tension and compression forces must have the relationship, $T = 2C$. This result can be formally obtained using the vector algebra discussed earlier in this chapter.

Acting on the left side of the frame, are two compression force vectors, $\mathbf{C} = -C$, one on the top, and one on the bottom, and one tension vector $\mathbf{T} = +T$. The balance condition, Equation 2.1, is met if the sum of all the force vectors acting on the left side frame, is zero:

$$\begin{aligned} \mathbf{C}_{top} + \mathbf{C}_{bottom} + \mathbf{T} &= 0 && \text{(now replace each vector by its separated form)} \\ (-C) + (-C) + (+T) &= 0 && \text{(now add } +2C \text{ to both sides)} \\ +T &= +2C \\ T &= 2C \end{aligned}$$

This relationship can also be obtained using the forces acting on the frame's right side.

G. Pulleys

Machines use pulley wheels to manipulate and exploit forces. Ideal pulley wheels are massless, and free to rotate without friction on an axle that is fixed to a bar. The rope (or belt) is also massless, and passes over the pulley, as shown in Figure 2.15, Both ends of the rope and bar are held fixed somewhere off to the left.

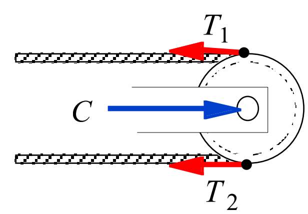

Figure 2.15 A pulley wheel and rope.

The tension in a rope might change when it passes over a pulley wheel as implied by the subscripts on the tension magnitudes. It may surprise you to learn that the magnitude of the tension forces in the top and bottom pieces of rope are the same! If the tension is higher on the top it will cause the pulley to rotate counter-clockwise. This brings rope from the bottom (increasing tension there) to the top (lowering the tension there). The turning of the pulley wheel tightens the bottom piece of rope and loosens the top piece of rope, thus equalizing the tension. This is represented by the equation, $T = T_1 = T_2$, or in words: the value of the two tensions, T_1 and T_2, are equal to each other and can be replaced in all their occurrences by the common value, T.

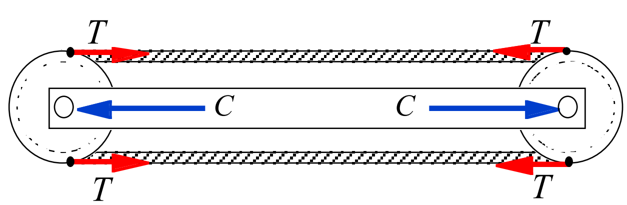

Figure 2.16 Pulley axles held apart by a compressed bar

The forces in Figure 2.15 could be generated to the left of the pulley by the situation shown in Figure 2.16. A belt runs over two pulleys held apart by a bar that supports the axles. The belt has a tension T, while the bar is compressed between the axles. This compression generates a force with a magnitude, $C = 2T$, on the axles, and they in turn push on each pulley to maintain the belt tension. If the top of the belt is grabbed in the middle and pulled to the right (and holding the bar in place), the tension forces on either side will become unequal briefly (the left side tension will increase and the right side tension will decrease). The pulleys turn and the belt tension equalizes. A step-by-step description of how the tension equalizes is left as an exercise.

You may have noticed that gravity did not play role in any of the examples shown. Gravity is a special case that must be handled carefully to avoid a few very common misconceptions about the balancing of forces. A number of future chapters will be devoted to gravity.

Chapter Summary:

- Force vector arrows are labeled with an italic letter, symbolizing the magnitude, and represented, approximately, by the arrow's length. To simplify force vector drawings, an object generating a force can be removed but a force must always be shown acting on an object.
- Force vectors are "equal", if, and only if, the magnitudes *and* directions are the same.
- External forces acting outward on a spring will stretch it, while external forces acting inward will compress it.
- Tension forces generated internal to a stretched spring pull inward on attached objects. Compression forces generated internal to a compressed spring, push outward on attached objects.
- Elastic objects will distort by an amount that is proportional to the applied force (Hooke's Law). A strong but lightweight (coil) spring is a good approximation to a massless ideal spring. A slinky toy is NOT an ideal spring.
- Two forces of equal magnitude and opposing directions (balanced forces), one at each end, are needed to maintain an ideal spring in stretched or compressed state. An attempt to create an imbalance in forces acting on a massless object, instead, will cause it to accelerate rapidly away.
- A vector, appears in print as a bold letter (e.g., **F**), or in script as a letter with an arrow above (e.g., $\vec{\mathrm{F}}$), has two parts, the magnitude, F (positive scalar), and direction; a sign (+ or –) specifies the direction in one dimension. Vectors in one dimension obey the standard rules of algebra.
- Force vectors acting on a single object are "balanced", if the *vector sum* of the forces is equal to zero: $\mathbf{F}_{\mathrm{net}} = \mathbf{F}_1 + \mathbf{F}_2 + ... + \mathbf{F}_n = 0$.

- Newton's 1st law: Balanced forces will not change the motion of an object with mass. This implies that with balanced forces acting on it, a stationary mass will NOT begin to move, and a moving mass will continue to move *at the same speed, in the same direction*.
- Balanced forces acting on an object generate internal forces of compression or tension.
- Newton's 3rd law (action-reaction): Acting at the boundary between objects are forces with equal magnitude and opposite direction; one force acting on each object. These forces do not "balance" because they do not act on the *same* object.