Chapter 3: Hooke’s law for ideal springs

A. Measuring the strength of a spring

An ideal spring, natural length, $L = 30$ cm, is placed next to a centimeter rule on a smooth and frictionless table, as shown in Figure 3.1. With no forces applied, the natural

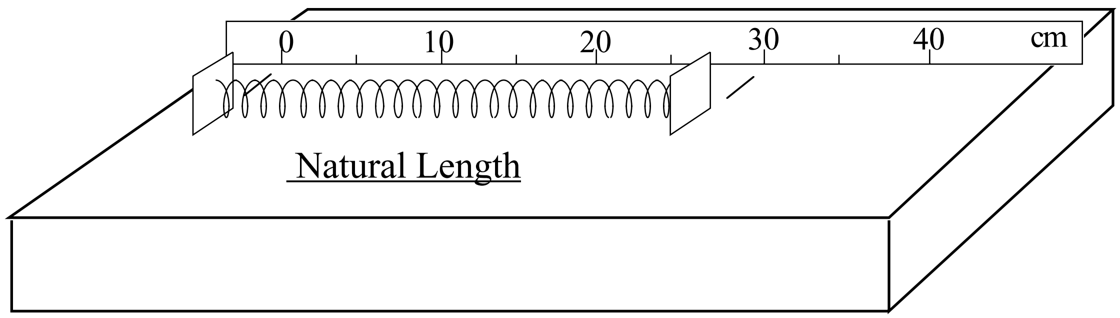

Figure 3.1 An ideal spring placed on a frictionless table.

length of the spring can be seen to measure 30 cm. While length changes are made to the spring, one end is held at 0 cm on the rule. For a given amount of stretch, if one end moves or both ends move, the same final forces must be applied.

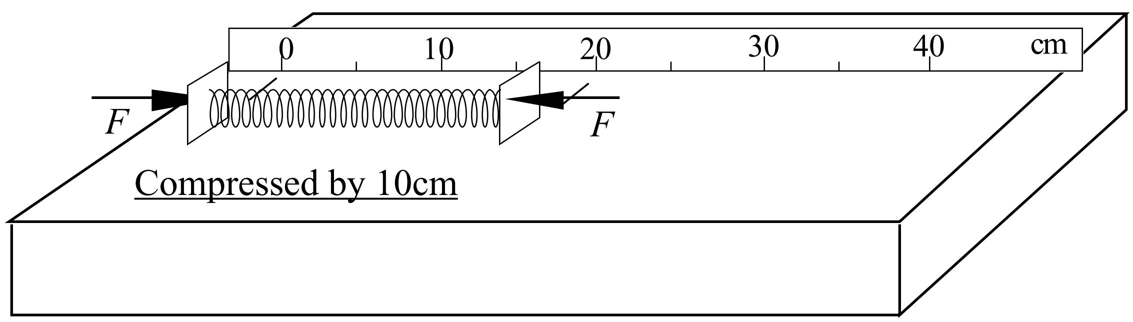

Figure 3.2 An ideal spring compressed by 10 cm.

Two forces must be applied, one on each end, of equal magnitude and opposite direction to change the length by amount x. The new length, l, is $l = L + x$ (if stretched), or $l = L - x$ (if compressed). As shown above in Figure 3.2, two inward facing forces, F, compress the spring by, $x = 10$ cm, resulting in the new length, $l = 20$ cm. Outward facing

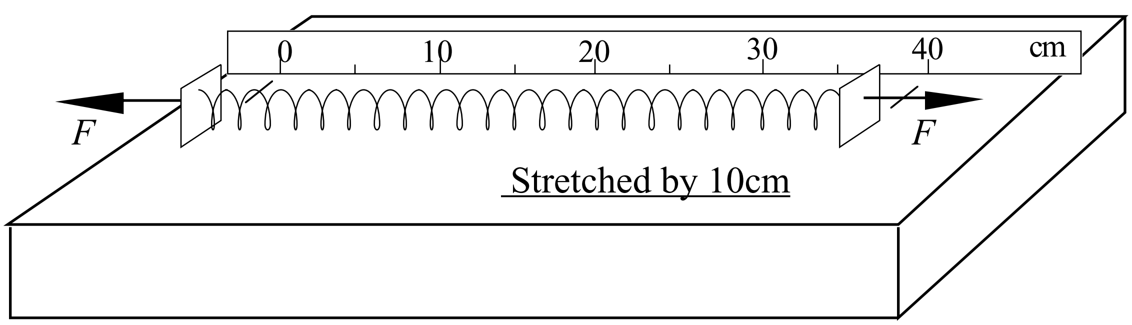

Figure 3.3 An ideal spring stretched by 10 cm.

forces of the same magnitude stretch the spring by $x = 10$ cm, to the new length, $l = 40$ cm, as shown above in Figure 3.3.

A force, F, applied to both ends of the spring, causes a 10 cm stretch. Increasing this force (on both sides) to a factor of 2 times the original force causes a 20 cm stretch. Increasing this force (on both sides) to a factor of 3 times the original force causes a 30 cm stretch, and so on. This data exhibits a linear relationship, between the force and stretch. The force applied divided by the size of the stretch is always 0.1 for this spring. The relationship stated verbally is "one tenth of a unit of force *per* centimeter of stretch", and written in a ratio as shown in Equation 3.2,

$$\frac{F}{x} = \left(\frac{1 \text{ unit of force}}{10 \text{ cm of stretch}}\right) = \left(\frac{2 \text{ units of force}}{20 \text{ cm of stretch}}\right) = 0.1\left(\frac{\text{units of force}}{\text{cm of stretch}}\right). \tag{3.2}$$

All springs with the same strength as the one being tested here will have this value for the ratio: 1/10 (unit of force)/(cm of stretch). A long and a short spring can have the same strength. The length of a spring is not the crucial parameter. The crucial parameter is the force needed to *change* the length by a given amount.

Another spring, twice as strong as the one above, takes 2 units of force to stretch it by 10 cm, 4 units of force to stretch it 20 cm, etc. This linear relationship stated verbally is "two tenths of a unit of force per centimeter of stretch", and written in a ratio as shown in the Equation 3.3, for this stronger spring.

$$\frac{F}{x} = \left(\frac{2 \text{ units of force}}{10 \text{ cm of stretch}}\right) = \left(\frac{4 \text{ units of force}}{20 \text{ cm of stretch}}\right) = 0.2\left(\frac{\text{units of force}}{\text{cm of stretch}}\right) \tag{3.3}$$

For a given spring, the ratio of the force applied and the amount of stretch is a constant. This constant is usually specified by the symbol k and is called the "spring constant". The spring constant defines the *strength* of the spring so that higher values of k mean stronger springs. A universal relationship between the force and the stretch is then:

$$\frac{F}{x} = k\,. \tag{3.4}$$

The first spring measured above had a spring constant k = 0.1 (units of force per cm of stretch) while the second spring had a spring constant k = 0.2 (units of force per cm of stretch. For stronger springs, k will have even larger values, for weaker springs smaller values; *spring constants cannot be zero or negative*.

If these springs are compressed the ratio of the magnitude, F, to amount of compression, x, is the same positive number k found when stretched. Equation 3.4 can be used for a stretched or compressed ideal spring. Multiplying both sides of Equation 3.4 by x, we can rewrite the relationship in the more common form of Hooke's law:

$$F = kx. \tag{3.5}$$

Equation 3.5 relates the force magnitude, F, applied to *both sides* of an elastic object with spring constant, k, causing a length change, x, from the natural length. Hooke's law will be considerably more useful when the unit of force, are defined in the next section using the metric system of units.

B. The newton unit of force.

In force units called newtons and length in meters, the value of k for a spring can be determined by oscillating a known mass, m (in kg), at the end of the spring and measuring the repeat time of the motion (the period), t, in seconds. It is always found in experiments, that the spring constant, its mass, and the period of motion are related by, $k = m / (t / 2\pi)^2$, (sinusoidal motion motivates the factor of 2π). This indicates that the units of force, distance, mass, and time are not independent!

The unit of force is defined with a *standard* spring chosen such that a 1 kg mass oscillates with a period, $t = 0.628$ s, and will be known as the "Newton spring". We place this standard spring in a special room in Paris. Next to the spring, there is a real cylinder of metal and a ruler, defining the standard 1 kilogram (1 kg) mass, and 1 m length. Any change in the standard mass or length generates a corresponding change in the standard spring to maintain the relationship found above.

The unit of force is then defined as two opposing forces each with a magnitude of $F = 1$ N, one at each end of a Newton spring, will change the length of the spring by 1 cm (0.01 m). Note that specific changes in length determine *two* opposing forces of equal magnitude, one at each end of the spring. A larger force, $F = 100$ N, (it is to be understood that two opposing forces of this magnitude are always needed) will stretch a Newton spring by 1 m (however, to remain elastic when stretched by 1 m, this spring must have a natural length of 2 m or more).

By making copies of a Newton spring and distributing them around the world, we can be sure that a force measured to be 1 N in Paris (1 cm of stretch) is the same force in the US, and everywhere else, even on a spaceship halfway to the next galaxy. How big is 1 N of force? It can be put into perspective by noting that 1 N is approximately the force you feel when holding at your local fast-food establishment, a 1/4-pound hamburger (0.22 lb., after cooking). This delicacy is not called a "1/4 pounder" in Paris, where you would have to ask for a "Royale"; or so we are told.

The units of k in Equation 3.5 can now be defined. Our standard spring, the Newton spring described above, has a spring constant, $k = 1\,\mathrm{N/cm}$, or in words "1 newton per centimeter". Spring constants have compound units that express the strength of the spring in terms of force and distance. Using Hooke's law (Equation 3.5), the force, F, that stretches a standard spring (spring constant k = 1N/cm) by a distance, $x = 10$ cm, is calculated as follows:

$$\begin{aligned} F &= kx \\ &= (1\ \text{N/cm})(10\ \text{cm}), \\ &= \underline{10\,\text{N}} \end{aligned}$$

where the units of the product, kx, are determined by the standard rules of algebra. The centimeter units of length in the denominator of k are canceled by centimeter units of the length, x, so that only the units of a force remain in the product.

This is not an accident: the units of k were constructed in such a way that this would happen. Another example is a car moving with a speed v (miles/hr) in a time t (hrs). It will travel a distance, $s = vt$, where the units (hr) in the denominator of v cancel with those of the time traveled, t, leaving only the units of a distance (miles) in the product.

C. Spring constants using Scientific Notation

Spring constants also characterize physical springs that do not have all the characteristics of an ideal spring. These springs vary in strength from very weak to very strong, and come in all shapes and sizes. Often, scientific notation is needed to specify a spring constant. Inside a mechanical watch, the very regular oscillations of a very weak coiled spring calibrate its second of time and cause a geared wheel to rotate, and the second hand to move. The spring constant of such a spring might be $k = 1\times10^{-3}\,\text{N/cm}$. Earlier it was pointed out that a 1N force is a weight of about 1/4 lb., therefore, 10^{-3} N (one thousandth of a Newton) can't be much more than the weight of a housefly and yet it would stretch the watch spring by 1 cm. This is a very delicate spring, requiring special tools to handle it without damage is so weak compared to its weight, that it cannot be considered as massless and will have characteristics closer to a slinky than to an ideal spring.

The strong spring attached to one wheel in the suspension of a car is compressed about 10 cm by a force of about 3000 N (each wheel holds up 1/4 of the car's weight). The spring constant is close to:

$$\begin{aligned} k &= F/x \\ &= (3000\text{N})/(10\text{cm}) \\ &= \underline{300\,\text{N}/\text{cm}} = 3\times10^{2}\,\text{N}/\text{cm}, \end{aligned}$$

and that is 300,000 times stronger than a watch spring. Heavier vehicles like a large truck might have springs that are 10-100 times stronger than a car, or as high as $3\times10^{4}\,\text{N/cm}$.

To compress such a spring by a significant amount, the forces needed are very much larger than the spring's weight. These springs can be considered "massless", in the sense that if one *tries* to apply a force of this magnitude to only one end of the spring, it will not compress, but instead, will move away rapidly in the direction of the comparatively small force that can actually be applied.

D. Elastic properties of materials

Instead of bending a steel rod into the shape of a coiled spring, take a small straight section and compress it along its length as shown in figure 3.4.

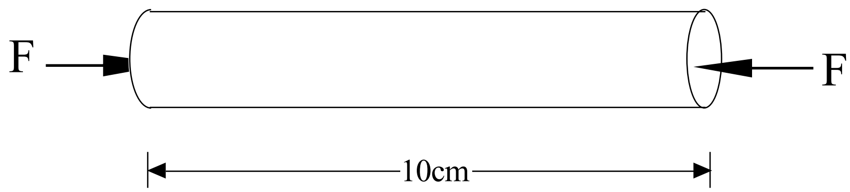

Figure 3.4 Steel bar 10 cm long being compressed by forces.

This is a very strong spring. Using the measured elastic properties of steel, it can be shown that this rod will have a spring constant of about $2\times10^{6}\,\mathrm{N/cm}$. If this rod is squeezed with a force equal to the weight of a large car, $F = 20{,}000\,\mathrm{N}$, the rod compresses by:

$$x = \frac{F}{k} = \frac{2\times10^{4}\,\mathrm{N}}{2\times10^{6}\,\mathrm{N/cm}}$$

$$= \underline{.01\text{ cm}}\ \ (\text{equivalents: } 0.1\text{mm},\ 100\,\mu\text{m} = 100\times10^{-6}\text{ m},\ \ 10^{-4}\text{ m}).$$

This is a small compression, about the thickness of a few sheets of paper, but it is observable and can be measured with standard tools (a micrometer, for example). Squeezing the rod generates compression forces that push back on each end with a reaction force, also 20,000N. Removing the applied forces allows the “spring” to expand by 0.01 cm, and return to its natural length, 10 cm. Though this very strong spring generates large reaction forces, one would scarcely notice the motion of the ends during the expansion.

This spring could not be used in devices where springs make an object move, as in a pinball machine. If the spring constant is very large, the generated forces will be eliminated, typically, by only microscopic length changes. The elastic properties of materials with large spring constants are often overlooked because the amount of stretch or compression is too small to be seen without magnification. Much confusion ensues, however, if their elastic forces are ignored.

There are other measures of material strength. The force at which an object looses its elastic properties is known as the “yield point”, and the force at which it will break is known as the “ultimate tensile strength”. Unless otherwise stated, when the strength of a spring is mentioned it will always refer to the spring constant. String, rope, and wire can have large spring constants. They produce useful tension forces when stretched, but break before the length change is very noticeable with the naked eye. In all other respects, they behave in the same manner as more stretchable springs.

E. Graphing Hooke's law.

Hooke's law is a relationship that can be displayed using a graph. The graph, shown below in Figure 3.5, displays the experimental measurements for two ideal springs.

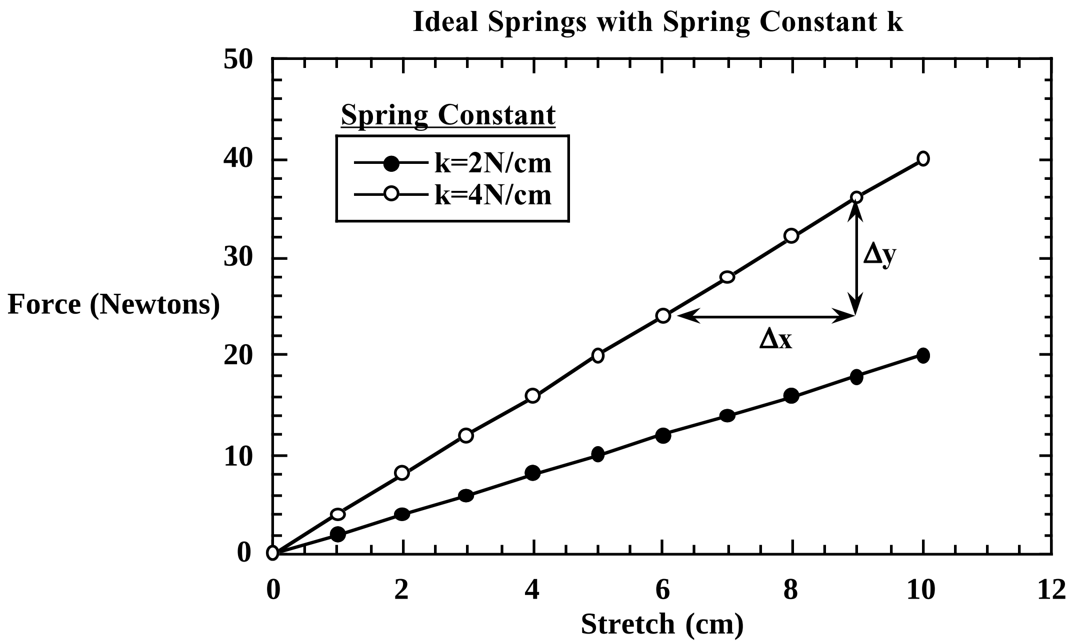

Figure 3.5 Plot of the force vs. the amount of stretch for two ideal springs

The springs were stretched by an amount x (shown on the horizontal, x-axis) from zero in one centimeter increments. The force F (shown on the vertical, y-axis) required to make that length change was recorded. One spring has a spring constant $k = 2\,\text{N/cm}$ (solid dots) and the other, twice as strong, has a spring constant $k = 4\,\text{N/cm}$ (open circles).

The data shown in the plot was used to determine the two spring constants. We know that an ideal spring will respond to an outward applied force F by stretching an amount x according to Hooke's law that is seen below to be in the form of the equation for a straight line:

$$F = kx \qquad \text{[Hooke's law]},$$

$$y = mx + b \qquad \text{[straight line]},$$

where the variable plotted on the vertical axis (y) is the force F, while the variable plotted on the horizontal axis (x) is the amount of stretch x. The slope of the line, m, multiplies x

in the straight line equation. In Hooke's law, k multiplies x, and therefore, the data for an ideal spring should lie on a line with slope $m = k$, and intercept $b = 0$.

Warning, do not make the mistake of assuming that the intercept b is zero when calculating the slope. There are springs that must first have an initial force, F_0, acting on it before following a linear relationship (a modified Hooke's law) between the additional applied force and the stretch, $F = kx + F_0$, and for this spring the line has an intercept, $b = F_0$. To determine the slope of a line, the procedure outlined below always gives the correct answer.

The slope one of the springs shown in Figure 3.5 is obtained by taking the ratio of the change in y (written Δy) for a given change in x (written Δx):

$$\Delta x = \left(x_2 - x_1\right) = \left(9 - 6\right)\text{cm} = 3\,\text{cm},$$

$$\Delta y = \left(y_2 - y_1\right) = \left(36 - 24\right)\text{N} = 12\,\text{N},$$

$$k = \frac{\Delta y}{\Delta x} = \frac{12\,\text{N}}{3\,\text{cm}} = 4\,\text{N/cm} \text{ (the slope, } m, \text{ in the equation of a line).}$$

Note that the slope of the line for a graph of the force, F, vs. the stretch, x, is a quantity with the units N/cm, consistent with the identification of the slope with the spring constant k.

It is important to recognize that the slope of a line can have units as well as a value. For example, a plot of the position (s) versus time (t) for an object traveling with a constant speed (v), is a straight line. The line has a slope $\Delta s/\Delta t$, the speed v with the proper units of a distance per unit time (e.g., meters/second).

F. Units conversion reference guide

Conversion of units is a process that must be mastered. If a conversion "factor" is well known, such as 60 minutes per hour (60 min./hr.), you must still decide if multiplication or division by the factor is required. It is helpful to consider the process of conversion of units as multiplication by 1. Any quantity multiplied by 1 will not change its value, *but it can be used to change its units*.

The definition of the prefix in the word *centi*meter tells us that 1 centimeter is one hundredth (1/100) of a meter, and therefore, there are 100 centimeters per meter. Turning the words "100 centimeters **per** meter" into a ratio (r):

$$r = \frac{100\,\mathrm{cm}}{1\,\mathrm{m}} = 100\,\mathrm{cm/m}.$$

The numerator and denominator of the fraction, r, are the same length, but in different units. The ratio must have a value exactly equal to 1. The unit-less quantity, 1, has been written as a ratio of two identical lengths that contain units of a needed conversion. A length value given in meters multiplied by r yields the same length in centimeters. The centimeter equivalent of, $x = 0.5\,\mathrm{m}$, is found by multiplying x by $r = 100\,\mathrm{cm/m}$:

$$x = rx$$

$$\left(100\frac{\mathrm{cm}}{\cancel{\mathrm{m}}}\right)\left(0.5\ \cancel{\mathrm{m}}\right) = 50\ \mathrm{cm},$$

where the meter units cancel leaving only the centimeter units as desired. The words "1 meter is equal to 100 centimeters" can also be turned into a ratio (r'), also equal to 1:

$$r' = \frac{1\ \mathrm{m}}{100\ \mathrm{cm}} = 0.01\frac{\mathrm{m}}{\mathrm{cm}},$$

and it can be used to make the conversion from centimeter units to meter units of distance by multiplication as before. Both r' and r are equal to 1 but they are expressed in different units and the two ratios are inverses of each other, $r' = 1/r$.

The conversion of a quantity called, k = 5N/cm, with the compound units of newtons per cm (N/cm) to the units, N/m, is accomplished by multiplying k by r:

$$k = 5\frac{\mathrm{N}}{\mathrm{cm}}$$

$$= rk = \left(\frac{100\ \cancel{\mathrm{cm}}}{1\ \mathrm{m}}\right)\left(5\frac{\mathrm{N}}{\cancel{\mathrm{cm}}}\right) = \underline{500\,\mathrm{N/m}}$$

where the centimeter units cancel, as desired, leaving only the N/m units. If by mistake you multiply instead by r', the centimeter units would not cancel, informing you of the error. Also, the answer should make sense to you: a spring that requires 5 N of force to stretch by 1 cm, must take more force (500 N) to stretch 100 cm (or 1 m), and so it does.

Chapter Summary:

- To change the length of an ideal spring, two forces must act, of equal magnitude and opposite direction, one on each end of the spring. A spring has its natural length when no forces act on it.
- A spring constant, k (units of a force divided by distance), specifies the strength of an ideal spring that stretches or compresses by a distance, x, when two forces, $F = kx$, are applied in opposite directions, one on each end of the spring.
- To use Hooke's law, $F = kx$, when k is given in units different from those of the distance, x, or force, F, a conversion of units will be required. Converting the units of a quantity is equivalent to a multiplication by a ratio, equal to one, that will introduce the desired units and eliminate the unwanted units (see the guide for units conversion, section H, above).
- A Newton spring has the spring constant $k = 1\,\text{N/cm}$ (or the equivalent, 100 N/m). If 1 N forces are applied in opposite directions to the ends of a Newton spring the length changes by 1 cm (0.01 m)
- A force of 1 N is about 0.22 lb.; easily remembered as close to the weight of a 1/4 lb. hamburger.
- Strong solid materials, such as steel or wood, have properties of a spring with a very large spring constant. In most situations, only the resulting force is important and not the value of the small distortion.
- If a material obeys Hooke's law for an ideal spring, $F = kx$, a graph of applied force, F, vs. distortion, x, yields a straight line with slope (units of a force/distance) equal to the spring constant, k.
- The slope of a straight line must be calculated using the standard formula and two points on the line. The point (0,0) can be used only if that point is shown on the line (see warning in section G of this chapter).