Chapter 8: Work and Energy

A. The definition of energy

Although related, the concepts of force and energy are not the same. It has been emphasized, particularly in chapter 4, that a stretched ideal spring *generates* forces and *stores* something else, called energy. This difference between force and energy is demonstrated by a simple example: if equal forces are applied to springs with very different spring constants, such as a rope and a bungie cord, much more energy is stored by the bungie cord than by the rope.

The energy stored by a distorted spring is called "potential" energy, PE_S, where the subscript, *S*, identifies the energy as stored by a spring. This can be restated as a definition: "energy is the quantity *stored* by a stretched or compressed spring". No other quantity has been found that can be stored by a spring.

Implicit in the definition is a process transferring energy to a spring from masses, and from a spring to other objects (masses). Over the time it takes for an ideal spring to return to its natural length, its stored energy is transferred to the kinetic (motion) energy of the masses at the ends of the spring. This results in an increase in the total "kinetic" energy (*KE*) of the masses. If the kinetic energy is retained by the masses, and not removed by another force, the change in their energies will be observed as a change in their speeds. The kinetic energy of an object is dependent on the observer's frame of reference, thus making it a poor place to start the discussion of energy.

A compressed ideal spring, on the other hand, will look the same to observers that are stationary or moving (slowly compared to light). In a transfer of a potential energy to kinetic energies of masses, moving and stationary observers performing careful measurements, agree on the potential energy that was stored by the spring. Gravity can also store a potential energy, PE_G, however, its discussion is complicated by the fact that the masses are involved in both potential and kinetic energies.

B. Storage of energy in a compressed spring

A weak spring with spring constant, k_{weak}, and very strong spring with a spring constant, $k_{strong} = 100k_{weak}$, compressed by the same force, *F*, will react as shown in Figure 8.1. The force of a weight, compresses the weak spring by a large distance, x_w, but the strong spring compresses by only a small amount, x_s. The relative compression can be obtained by an application of Hooke's law:

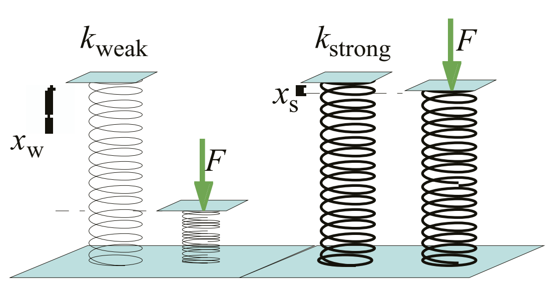

Figure 8.1 A weak spring (left) and a strong spring (right) compressed by the same force.

$$\text{(weak spring) } x_w = \frac{F}{k_{weak}}$$

$$\text{(strong spring) } x_s = \frac{F}{k_{strong}} = \frac{F}{100\, k_{weak}}$$

$$= \tfrac{1}{100}\left(\frac{F}{k_{weak}}\right) = \tfrac{1}{100} x_w \ .$$

The weak spring will compress a distance 100 times further than strong spring. The force must act over a large distance, x_w, to compress the weak spring, while the force acts over only a small distance on the strong spring. Stronger springs, such as a cylinder of metal, compress even less. A large weight on a strong spring will barely budge it. Compressing the weak spring with the same weight, however, can do some real damage. The relative effect of the same force acting on these two springs reflects a difference in the amount of energy stored by the springs.

A wood block will compress microscopically using only the force of our hands. Compressing a wood block with our hands stores much less energy than is stored by compressing a bedspring. The bedspring is compressed by an observable distance, and has the capability of pushing our hands apart for this longer distance. The compressed wood block, however, is capable of pushing our hands apart by only a microscopic distance, before the force it applies to our hands drops to zero.

A wood block, compressed as hard as possible with a hand on either side, will be quite stable. Trying to hold a bedspring compressed in the same way results in a shaking motion that is difficult to control. The compressed bedspring reacts to any small reduction in the applied force by expanding against the hand masses. This causes energy to be dissipated as heat in our muscles. The muscles must then do additional work to re-compress the spring. Small changes in the applied force results in observable length changes of the bedspring. Significant amounts of energy are transferred into and out of the spring during the resulting vibrations, much more than in any variations of force on the wood block.

C. Work and Potential Energy

Humans do work using chemical potential energy stored in molecules of the food that we eat. Without this source of energy, anything living thing would soon run out of the energy necessary to sustain life. The energy obtained from food originated in the sun (fusion reactions), arrived on earth via sunlight and was stored in organic molecules by chemical reactions (photosynthesis) in plants. These organic processes are very complicated, but their essence is contained in the following analysis of a force doing work on the mass at the end of a spring and thus storing energy in the spring.

An external force does work on a mass if the mass moves in a direction parallel to the force (for a force that changes during the motion, it is the average force over this distance that counts). If an object does not move under the action of an applied force, that force does no work on the mass.

The distance and direction moved are combined into a vector that is known as the displacement vector, **s** (the symbol **s** is used, not *x*, to emphasize the need for the direction as well as the distance moved). If the applied force and displacement vectors point in the same direction ($\mathbf{F} = +F$ & $\mathbf{s} = +s$, or $\mathbf{F} = -F$ & $\mathbf{s} = -s$) then the work done, w, by that force is the product of the magnitudes of the two vectors, the force and the displacement, $w = Fs$, and the work done is positive. If the two vectors point in opposite directions ($\mathbf{F} = \pm F$ & $\mathbf{s} = \mp s$) then the work done is negative, $w = -Fs$. The units of work are the same as the units of a force times a distance. With the force given in newtons and the distance in meters, work has the units, N·m.

When a spring is slowly stretched or compressed, at the spring ends external forces do "positive" work on the masses, and transfer energy from the source of the external force to the kinetic energy of the masses. At the same time, spring forces do "negative" work on each mass, transferring kinetic energy of the masses to the potential energy stored by the spring. The masses at the ends of the spring act as carriers of energy but never attain a significant kinetic energy. Under these conditions, the work by external forces can be considered as done directly on the spring, though in reality, the external forces and internal forces do work on the masses at the ends of the spring.

An examination of the signs of the work done on a spring by external forces is shown in Fig. 8.2. The left end of the spring is held at a fixed location so that the distance the right moves is equal to the change in the spring's length. An *inward* force is required to move the end of the top spring *inward* during a compression. The force must continuously increase as the compression proceeds but at all times the force (F) and the displacement (s) are in the same direction. The work done by this force is therefore positive, adding potential energy to the spring. Stretching the spring, as shown at

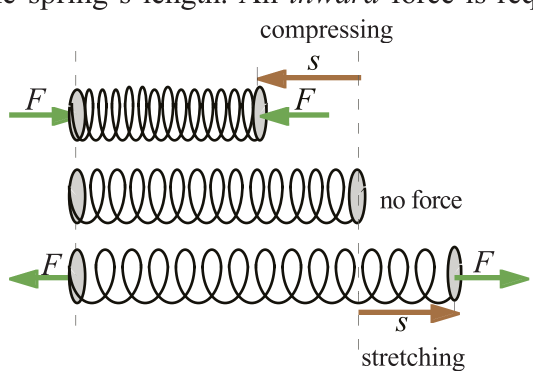

Figure 8.2 Force and displacement vectors compressing (top) and stretching (bottom) a spring. Dashed lines indicate the natural length of the spring.

the bottom of Fig. 8.2, requires an *outward* force while the end of the spring travels *outward*. Again, the force and displacement are in the same direction, and the work done by the force is positive, adding potential energy to the spring.

Now that the spring is either stretched or compressed consider the displacements when the external forces are reduced slowly to zero as shown in Figure 8.3. Allowed to slowly contract, or expand the spring's reaction forces do positive work on the masses and increase their kinetic energies while external forces do negative work and receive energy from the masses. If the spring's length does not change, regardless of any shift of the spring from one point to another, no energy transfer occurs between the masses and the spring.

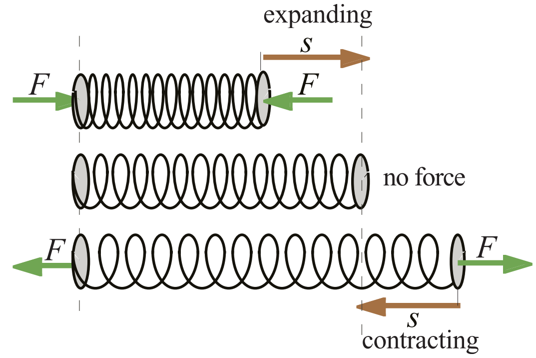

Figure 8.3 Force and displacement vectors in an expansion (top) and contraction (bottom). Dashed lines indicate the natural length of the spring.

Though these processes are assumed to occur slowly, it is not possible to remove compressing forces (e.g., generated by your hands) so quickly from an ideal spring that the spring cannot keep up with their movement. A massless ideal spring can expand infinitely fast and its ends will always be able to keep up with the source of a decreasing external force (e.g., your hands have mass and cannot be pulled away from a massless spring faster than the spring can expand). The objects that generate the external forces are always in contact with the spring ends.

When done slowly, there is always an exact equivalence between the work done by external forces on the masses and the potential energy stored in the spring. Energy "flows" into a spring when it is stretched or compressed. The energy that flowed into a spring from the exterior world is returned to the external world when it expands or contracts to its natural length.

D. Conservative forces.

Forces that can store energy in the manner of an ideal spring described above and then return the same amount are known as conservative forces. Gravity and electric forces are both conservative forces. The forces generated by an ideal spring are conservative because the ideal electric forces between atoms are themselves conservative forces. Real springs can come close to this ideal; however, a cycle of distortion and relaxation will always transform some of the mechanical (potential and kinetic) energy into another form called heat. Heat cannot be treated as a potential energy (heat energy is the net kinetic or vibration energy of the molecules in an object. The average *KE* of the molecules is proportional to the temperature of an object).

Static frictional forces prevent an object from moving, and therefore no energy transfer occurs. Sliding friction is the most important example of a non-conservative force. Frictional forces on objects sliding against each other distribute some or all of their kinetic energy into heat energy. Distributed as heat energy (kinetic energy of a large number of atoms), only a fraction of this energy can be used at a later time to compress or stretch a spring, the rest must remain as heat. The irreversible transfer of energy in a cycle involving heat is known as the 2nd law of thermodynamics, and prevents the construction of a perpetual motion machine. For our purposes just remember that sliding friction is a non-conservative force: it does not store energy as a potential energy that can be fully recovered at a later time.

E. Potential energy of a distorted spring.

As defined earlier, work done by a force is the product of the magnitude of the force vector and the magnitude of the displacement vector with a sign that depends on the relative direction of the two. If we define a way to multiply vectors, called the dot product, a simple formula for work can be written:

$$\text{Work} = \mathbf{F} \bullet \mathbf{s}, \tag{8.1}$$

the work done by a force is the dot product of the force vector (**F**) and the displacement vector (**s**). Though work can be either positive or negative (like credits and debits to a bank account), work is a scalar. The multiplication in Equation 8.1 for vectors in one dimension is handled with the standard rules of algebra. A force vector, $\mathbf{F} = +F$, and a displacement vector in the opposite direction, $\mathbf{s} = -s$, results in the scalar quantity work, $w = \mathbf{F} \bullet \mathbf{s} = (+F) \bullet (-s) = -Fs$, that is negative.

During the stretch of a spring with the left end held fixed, as shown in Figure 8.4, the applied force starts at zero, increases according to Hooke's law, reaches $F/2$ after half the stretch and the final value, F, at the full stretch. The average value of the force acting over the distance, x, is half the final value achieved when the spring is fully stretched. The work

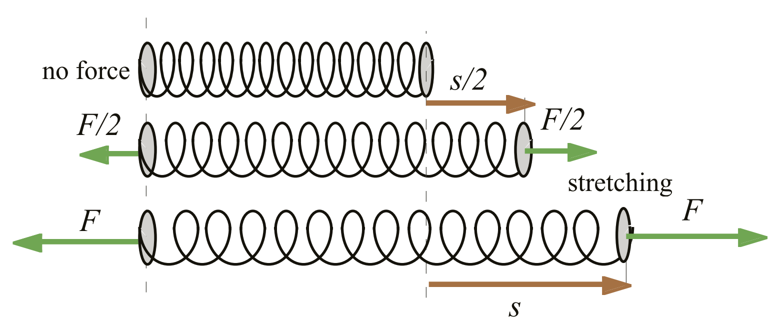

Figure 8.4 Stretching a spring in two steps.

done by the force, **F**, during the compression is calculated using the average force, $\langle \mathbf{F} \rangle = +F/2 = +kx/2$, and the full displacement, $\mathbf{s} = +x$, in Equation 8.1:

$$\begin{aligned} w &= \mathbf{F} \cdot \mathbf{s} = \langle \mathbf{F} \rangle \cdot \mathbf{s} \\ &= \left(+\tfrac{1}{2}kx\right) \cdot (+x) = \underline{\tfrac{1}{2}kx^2} \ . \end{aligned} \tag{8.2}$$

The applied force does work and transfers, through the mass, that amount of energy to the spring, to be stored as a potential energy with the same value:

$$\boxed{PE_S = \tfrac{1}{2}kx^2} \tag{8.3}$$

and the same units $(\text{N/m})(\text{m})^2 = \text{N} \cdot \text{m}$. A new unit is defined for both: the joule (J) with 1J = 1 N·m.

G. Energy in series and parallel springs

Each of the two identical springs compressed by the same force, F, a distance, x, as shown in Figure 8.5; will contain a potential energy $PE_S = \frac{1}{2}kx^2$. Joined in series or in parallel, as discussed in an earlier chapter, the total potential energy will be $2PE_S$ for both pairs.

When the springs are joined, the forces required to maintain each spring at the original compression differ: the same force, F, is needed in series, but a force, $2F$, is required in parallel. The compression of the pairs, on the other hand, is $2x$ in series while in parallel it remains x. Using Equation 8.1 ($<\mathbf{F}>$ is the average force acting during the compression), the work done in compressing the two combinations does yield the same value for both:

$$\text{(series)} \ w = \langle \mathbf{F} \rangle \bullet \mathbf{s} = \left[\tfrac{1}{2}(+F)\right](+2x) = Fx,$$

$$\text{(parallel)} \ w = \langle \mathbf{F} \rangle \bullet \mathbf{s} = \left[\tfrac{1}{2}(+2F)\right](+x) = Fx.$$

When the two springs are connected in series or in parallel, and each spring is compressed by the same amount, the same potential energy is stored, while the forces required to accomplish the compression differ by a factor of 2; a reminder that springs store energy and not forces.

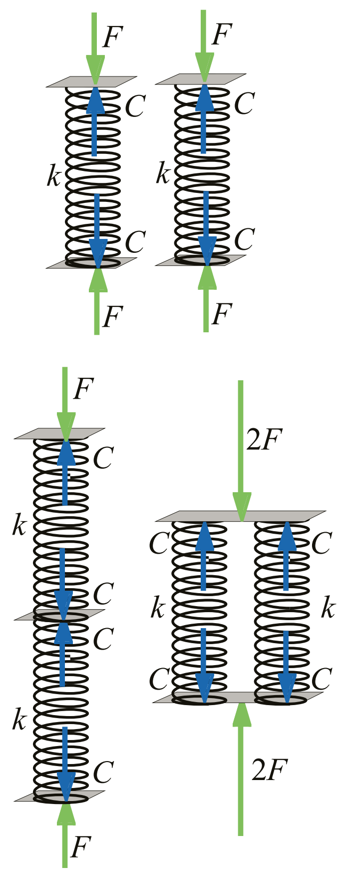

Figure 8.5 (top) Two compressed springs, (bottom) attached in series (left) and in parallel (right).

Chapter Summary:

- If the same force acts on springs with different spring constants, more energy is stored in the weaker spring (typically long) than in the stronger spring. Objects with very large spring constants can typically store only small amounts of energy.
- A vector that specifies a distance and a direction moved is called displacement, **s**.
- Work on a mass during a displacement is a scalar: the "dot product" of the net force acting on the mass and the displacement vector of that mass, work = $\mathbf{F} \bullet \mathbf{s}$. For forces varying with position, the net force is averaged over the displacement.
- External forces slowly increased, stretch or compress a spring, and do work (positive) adding potential energy to the spring.
- External forces slowly reduced, allow a stretched or compressed spring to relax toward the natural length, and do work (negative) removing potential energy from the spring.
- A constant force, $\mathbf{F} = \pm F$, acting through a displacement, $\mathbf{s} = \pm s$, in the same direction does an amount of work, $w = \mathbf{F} \bullet \mathbf{s} = Fs$.
- A constant force, $\mathbf{F} = \pm F$, acting through a displacement, $\mathbf{s} = \mp s$, (opposite directions) does an amount of work, $w = \mathbf{F} \bullet \mathbf{s} = -Fs$.
- Potential Energy, $PE_S = \frac{1}{2}kx^2$, is stored by a spring, when stretched or compressed.
- Joules are the units of potential energy and work ($1\text{J} = 1\text{N} \cdot \text{m}$).
- The units of x must be in meters and the units of the spring constant k must be N/m to obtain PE_S in joules (units of k, given in units, N/cm, will require a conversion).