Chapter 11: Momentum and Energy

A. Two masses accelerated by a compressed spring.

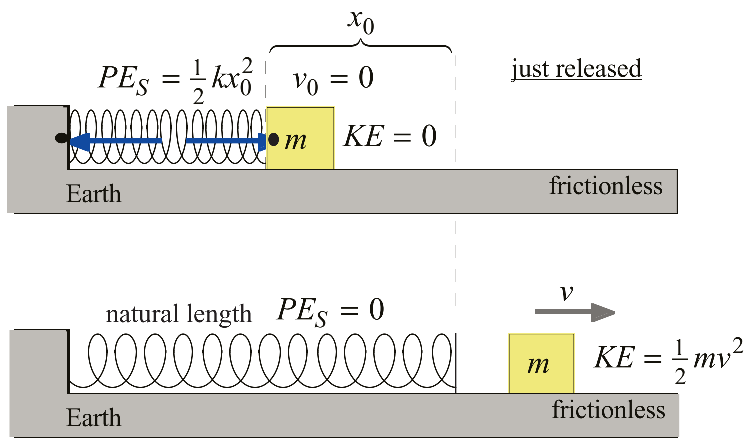

Figure 11.1 The potential energy of a compressed spring transferred to a single mass after release.

The transfer of potential energy stored in a spring to the kinetic energy of two masses is the starting point for the discussion of another property of motion called momentum. As described in chapter 9 and shown in Figure 11.1, a compressed spring expanding against a small mass, m, transfers all potential energy stored by the spring to the kinetic energy of the mass. There is a complete transfer of energy because the spring is attached to the Earth's huge mass (an extreme case) that does not move during the expansion. As can be determined by equating the initial potential energy, $PE_0 = \frac{1}{2}kx_0^2$, to the final kinetic energy, $KE = \frac{1}{2}mv^2$, the mass attains a final velocity, $v = \sqrt{\frac{k}{m}}x_0$.

Note that in comparing a spring's potential energy and the kinetic energy of the masses both must be measured in joule units, using the equivalence, $1\text{ J} = 1\text{ N}\cdot\text{m} = 1\text{ kg}\cdot\text{m}^2/\text{s}^2$.

The next example has identical masses, m, at each end of the compressed spring, as shown in Figure 11.2. As the spring expands, the spring's potential energy will be transferred in equal portions to the kinetic energy of the two masses. Each mass must receive half the initial potential energy, and therefore the speed u must be the same for

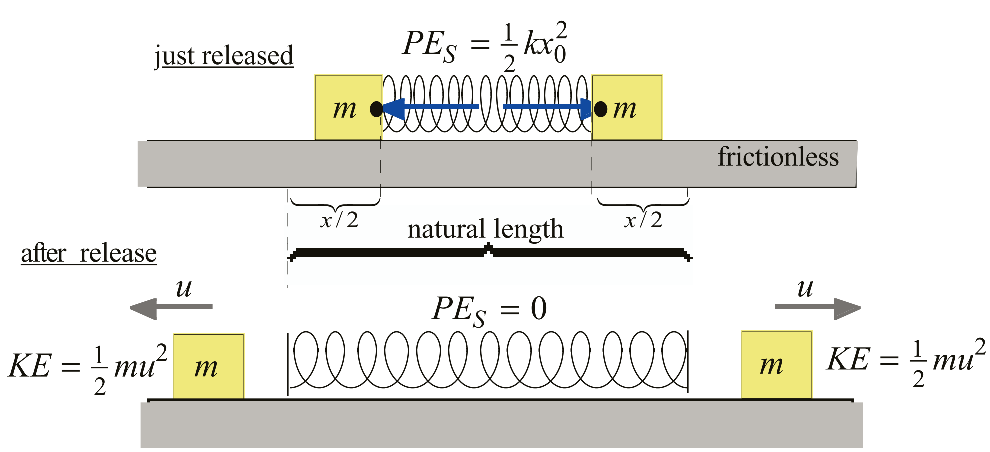

Figure 11.2 The potential energy of a compressed spring being split equally between two equal masses.

each mass. By equating the initial potential energy, $PE_0 = \frac{1}{2}kx_0^2$, to the sum of the final kinetic energies, $KE = \frac{1}{2}mu^2 + \frac{1}{2}mu^2$, one finds that each mass has a final speed, $u = \frac{1}{\sqrt{2}}\sqrt{\frac{k}{m}}x_0$. Comparing this speed, with the speed, $v = \sqrt{\frac{k}{m}}x_0$, when one mass takes all the energy, as shown in Figure 11.1, the speed, u, is lower by not a factor of two, but by only a factor of $\sqrt{2}$. This occurs because kinetic energy is quadratic in the speed.

Between the two extremes shown in Figures 11.1 and 11.2, there is a third case, shown in Figure 11.3, where m_1 is greater than m_2, but much smaller than the mass of the earth. All energy conservation can say is that the larger of the two masses, m_1, will have a speed, v_1, that is smaller than the speed, v_2, of the smaller mass. The speeds of any given masses cannot be predicted without some other constraint on the motion. Another

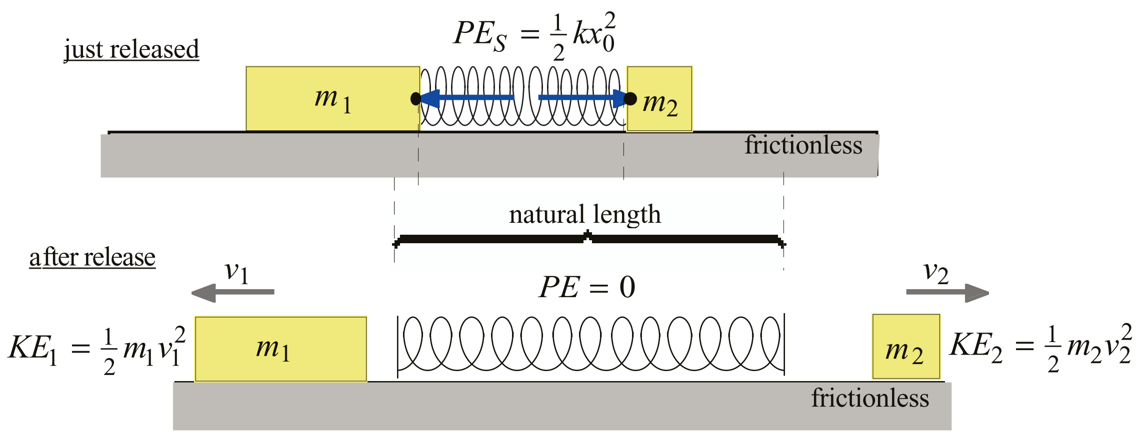

Figure 11.3 The potential energy of the compressed spring splits unequally between two unequal masses.

quantity associated with the motion of a mass fixes the ratio of the speeds. Newton called this quantity momentum.

B. Momentum

In the expansion of the spring, as shown in Figure 11.3, the same force magnitude (with average $\langle F \rangle$) acts on each mass. For each mass however, the displacement, s, the work done, and the kinetic energy change, $w = \Delta KE = \langle F \rangle s$, are clear in only the two extreme cases ($m_1 >> m_2$, or, $m_1 = m_2$) discussed above. From results of crude experiments similar to the one shown in Figure 11.3, Newton realized that during any time interval, Δt, $\langle F \rangle \Delta t$ would be the same for both masses. The quantity $\langle F \rangle \Delta t$ is called impulse and it causes changes in a quantity called momentum, p, such that $\Delta p = \langle F \rangle \Delta t$. Since force is a vector, so are changes in momentum, $\Delta \mathbf{p} = \mathbf{F}\Delta t$.

Both masses loose contact with the spring at the same time, Δt, after the release. At any time during the expansion, the compression force of the ideal spring acts on the masses in opposite directions with equal magnitudes. The change in the momentum of each mass is, $\Delta\mathbf{p}_2 = \langle +F\rangle\Delta t$, and $\Delta\mathbf{p}_1 = \langle -F\rangle\Delta t$, and therefore, $\Delta\mathbf{p}_2 = -\Delta\mathbf{p}_1$. The change in the momentum vectors of the two masses during the expansion will have equal magnitudes but opposite directions.

It is somewhat involved (see box below, if you are interested) to show that the momentum of a mass, m, traveling with a velocity, **v,** is

$$\boxed{\mathbf{p} = m\mathbf{v}} \qquad (p = mv) \qquad (11.1)$$

The direction of the momentum, **p**, is the same as the direction of the velocity, **v**. Velocity is a vector with a magnitude, v (speed), and a sign, (±), the direction of motion.

The expression for the momentum of a mass in terms of the mass, m, and speed, v, can be obtained by analyzing the effects of a constant force, **F**, acting for a time interval $\Delta t = (t - t_0)$, over a displacement, **s**, as shown in Figure 11.4. The velocity change $\Delta\mathbf{v} = (\mathbf{v} - \mathbf{v}_0)$, and the average velocity, $\langle\mathbf{v}\rangle = \dfrac{\mathbf{s}}{\Delta t}$, are used in the analysis of the momentum change, $\Delta\mathbf{p}$, of the mass:

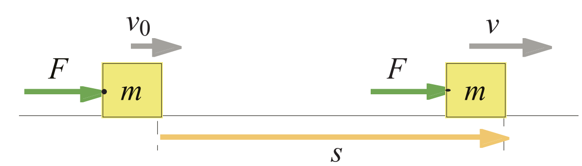

Figure 11.4 A mass with increasing speed caused by the action of constant force acting over a distance.

$$\frac{\Delta\mathbf{p}}{\Delta t} = \mathbf{F} \qquad \text{using}: \Delta\mathbf{p} = \mathbf{F}\Delta t$$

$$\frac{\Delta\mathbf{p}\cdot\mathbf{s}}{\Delta t} = \mathbf{F}\cdot\mathbf{s} \qquad \text{using}: \mathbf{F}\cdot\mathbf{s} = w = \Delta KE = \tfrac{1}{2}m(v^2 - v_0^2) = m\left[\Delta\mathbf{v}\cdot\frac{(\mathbf{v}+\mathbf{v}_0)}{2}\right]$$

$$\Delta\mathbf{p}\cdot\langle\mathbf{v}\rangle = m\Delta\mathbf{v}\cdot\langle\mathbf{v}\rangle \qquad \text{using}: \langle\mathbf{v}\rangle = \frac{\mathbf{s}}{\Delta t} = \frac{\mathbf{v}+\mathbf{v}_0}{2}$$

$$\Delta\mathbf{p} = m\Delta\mathbf{v}\,.$$

C. Momentum conservation

The constraint that the momentum changes of the two masses, shown in Figure 11.3, are equal in magnitude but opposite in direction, is equivalent to the statement that the VECTOR sum of their momenta, $\mathbf{p}_{net} = \mathbf{p}_1 + \mathbf{p}_2$, at the time of release, during and after the expansion, must remain the same. The momentum of each mass was zero at the start of the expansion, however, the momentum vector of each mass is no longer zero after the expansion, yet the net momentum, $\mathbf{p}_{net}$, a *vector*, will remain zero.

The generalization of this observation is known as the *conservation of momentum*: the vector sum of the momenta, the net momentum, $\mathbf{p}_{net}$, does not change during the interactions between the masses,

$$\mathbf{p}_{net} = \mathbf{p}_1 + \mathbf{p}_2 + \ldots + \mathbf{p}_n \text{ is constant,} \quad (11.2)$$

provided that there are no "external forces ($\mathbf{F}_{ext} = 0$)" acting on masses. An "external force" is any force that acts on the masses but its equal and opposite partner does not. The action of external forces will cause changes in the net momentum.

The conservation of momentum, Equation 11.2, applies in all interactions of masses even if energy is not conserved. Sliding friction, for example, is a non-conservative force and its action at the interface between two masses will reduce the available kinetic energy but momentum of the two masses will be conserved.

A bullet, fired from a gun, is given a large amount of energy by the explosion that occurs behind it in the barrel of the gun. The gun recoils with a momentum equal to the momentum of the bullet as it exits the gun barrel, but very little of the energy of the explosion is transferred to the gun. This is the problem of the speeds of unequal masses after the expansion of a spring (a well-controlled explosion), that can now be completely solved using momentum conservation.

D. Using momentum conservation

The complete transfer of the initial potential energy stored in the spring, PE_0, to the kinetic energies of the two masses following the expansion, as shown in Figure 11.5, is a consequence of energy conservation:

$$PE_0 = KE_1 + KE_2. \quad (11.4)$$

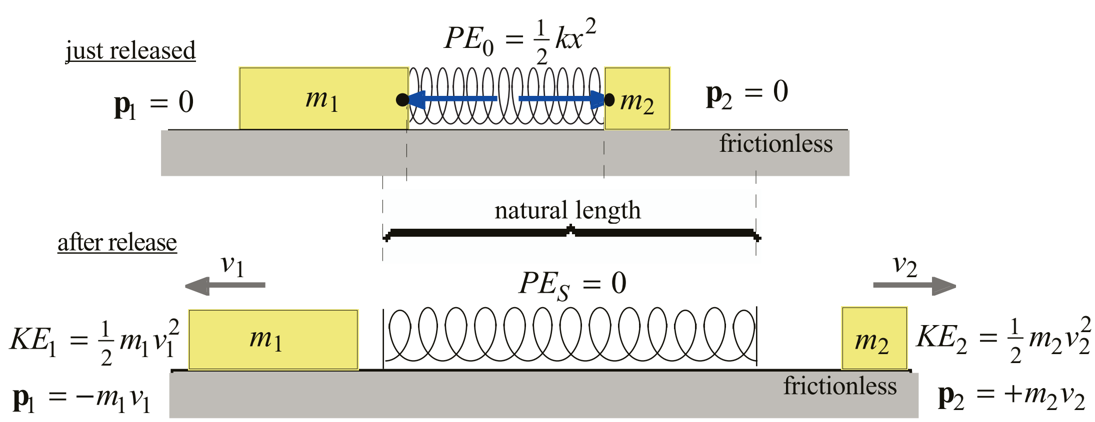

Figure 11.5 The transfer of the potential energy of a compressed spring to two unequal masses.

The momentum of each mass at the time of release is zero. After the expansion the velocities are, $\mathbf{v}_1 = -v_1$ and $\mathbf{v}_2 = +v_2$, where the small mass (m_2) is moving in the positive x-direction while the large mass (m_1) is moving in the negative x-direction. The corresponding momentum vectors are, $\mathbf{p}_1 = m_1 \mathbf{v}_1 = -m_1 v_1$ and, $\mathbf{p}_2 = m_2 \mathbf{v}_2 = +m_2 v_2$.

The initial momentum of each mass is zero, and so is the initial value of the net momentum:

$$\text{(just released)}\ \mathbf{p}_{net} = (\mathbf{p}_1 + \mathbf{p}_2)_0 = 0\ .$$

The net momentum after the expansion must also have this value:

$$\text{(after release)}\ \mathbf{p}_{net} = \mathbf{p}_1 + \mathbf{p}_2 = 0$$
$$= -m_1 v_1 + m_2 v_2 = 0\,.$$

Solving for the speed of mass 1,

$$m_1 v_1 = m_2 v_2 \qquad (11.5)$$

$$v_1 = \frac{m_2}{m_1} v_2, \quad \text{or} \quad \frac{v_1}{v_2} = \frac{m_2}{m_1} \qquad (11.6)$$

The speeds of the two masses in the extreme cases, where energy conservation was sufficient to determine them, are consistent with the values determined using Equation 11.6.

The case of unequal masses can now be fully solved in just a few steps of algebra. Squaring both sides of Equation 11.5 and removing one factor of m from each, will yield:

$$m_1\left(m_1 v_1^2\right) = m_2\left(m_2 v_2^2\right)$$

$$\left(m_1 v_1^2\right) = \frac{m_2}{m_1}\left(m_2 v_2^2\right). \qquad (11.7)$$

The terms in the brackets of Equation 11.7, apart from a factor of 1/2, are just the kinetic energies of the two masses,

$$KE_1 = \frac{m_2}{m_1} KE_2. \qquad (11.8)$$

Substituting the left side of Equation 11.8 into Equation 11.4, yields,

$$PE_0 = KE_1 + KE_2 \quad \text{(Eq. 11.4)}$$

$$PE_0 = \frac{m_2}{m_1} KE_2 + KE_2 = KE_2\left[1 + \frac{m_2}{m_1}\right]$$

$$KE_2 = \frac{PE_0}{\left[1 + \frac{m_2}{m_1}\right]} \qquad (11.9)$$

If the mass, m_1, is very large (like the mass of the earth) then the term in the brackets in Equation 11.9 is equal to 1 and tells us that the potential energy of the spring is transferred entirely to the kinetic energy small mass (m_2). If the masses are equal, the term in the brackets has the value 2. This means that the potential energy in the spring is split equally between the kinetic energies of two equal masses.

Using conservation of energy and momentum, the kinetic energies (and therefore the speeds) are determined for two known masses of any size propelled by a spring storing a known amount of potential energy. Momentum conservation alone can also explain many simple phenomena, as in the homework, that otherwise seem quite mysterious.

Return again to the process that started this chapter: the expansion of a spring, attached to the earth, against a small mass. After the expansion, to conserve momentum, the earth must have a NON-zero momentum, but the potential energy of the spring goes entirely into the *KE* of the small mass. Though the speed of the earth must be essentially zero, surprisingly, its momentum is not. The square of the momentum of mass 1, using the relationship between KE_1 and KE_2 of Equation 11.8, can be related to the kinetic energy of mass 2:

$$(p_1)^2 = (m_1 v_1)^2 = m_1 (m_1 v_1^2) = 2m_1 KE_1$$
$$= 2m_1 \left[\frac{m_2}{m_1}\right] KE_2 = 2m_2 KE_2 \ .$$

Though the speed of the earth is very small in this process, the cancellation of its mass from the expression for its momentum is a consequence of momentum conservation.

E. Time reversal and collisions.

Combining the steps where energy is first transferred to potential energy and then back to kinetic energy yields the classic "collision problem". For example, a small mass with some kinetic energy hits a wall and bounces off with the same energy it had before the collision. A collision, such as this one, where there is no change in the total energy of the masses is called an "elastic collision". Elastic collisions occur when the contact surfaces of the colliding objects compress and then expand, elastically, like an ideal spring. As discussed at the end of the last section, this collision may at first glance seem not to conserve momentum, however, the momentum of the earth is not zero, despite the fact, that the kinetic energy of the earth is zero.

A collision where energy is either lost or gained can be modeled by assuming that the energy difference is stored by a spring that doesn't fully expand or is partially compressed prior to the collision. "Collisions" where an amount of energy is released by the expansion of a compressed spring can, for example, model an explosion.

Underlying this discussion is the requirement that there are no external forces acting on the masses in the problem, and therefore, momentum will be conserved in all interactions between masses. Note also that a motion picture of an elastic collision run backward will look like an elastic collision with the directions of each of the masses reversed from the original. The collisions don’t distinguish one direction of time from another.

The same is not true for inelastic collisions. Place some clay on one of the masses so that it absorbs energy in the collision by being squashed. It would look rather odd in the motion picture run backward where the clay suddenly becomes undistorted and gives energy back to the masses. We know that clay just doesn't do that! Non-conservative forces are involved in making the permanent distortion of the clay and their actions are irreversible. This feature of non-conservative forces is related to the 2nd law of thermodynamics that then defines the true direction of time.

Chapter Summary

- The units of work, potential energy, and kinetic energy are joules ($1\ \text{J} = 1\ \text{N} \cdot \text{m} = 1\ \text{kg} \cdot \text{m}^2 / s^2$).
- The momentum of an object with mass, *m*, and velocity, **v**, is, $\mathbf{p} = m\mathbf{v}$
- Forces come in matched pairs! A force is "internal" to a system of masses if both members of the force pair act on those masses. A force is “external” to a system of masses if it acts in that system but its equal and opposite partner does not.
- The statement of momentum conservation is for *n* masses:
- $$\mathbf{p}_{\text{net}} = \mathbf{p}_1 + \mathbf{p}_2 + \cdots + \mathbf{p}_n \text{ is constant,}$$
  provided that there are no external forces ($\mathbf{F}_{\text{ext}} = 0$) acting on masses.
- A specific case: a system of masses and springs has an initial value of $\mathbf{p}_{\text{net}} = 0$. Provided that $\mathbf{F}_{\text{ext}} = 0$, the value of the net momentum vector for the masses will be zero at all times, though some or all of the masses may have non-zero momenta.
- The use of momentum conservation (when no external forces are acting) and energy conservation (if only conservative forces are acting) are powerful tools in the explanation of motion.
- The presence of external forces is indicated if momentum conservation is violated. The presence of non-conservative forces is indicated if energy conservation is violated. A violation of energy conservation does not demand that momentum conservation be also violated.