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PHY-852: QUANTUM MECHANICS II

Midterm, 30 points

March 18, 2002

PROBLEM 1. /8/ A wave function of a particle is

$$\psi(\mathbf{r}) = Axyz e^{-\beta r}. \quad (1)$$

Determine the probabilities of various values of l , m , and parity in this state.

PROBLEM 2. /6/ For a spinless particle in a three-dimensional isotropic potential $U(r)$,

- a. is it possible to have a level with the degeneracies $d = 2, d = 7, d = 9$?
- b. what is the possible degeneracy of the first excited level?

PROBLEM 3. /16/ a. Determine the scattering cross section for a low energy particle from a repulsive potential

$$U(r) = \begin{cases} U_0 > 0, & r < R, \\ 0, & r > R. \end{cases} \quad (2)$$

- b. Check your answer in the limit $U_0 \rightarrow \infty$.
c. Compare the result (a) with the Born approximation.

SOLUTIONS

PROBLEM 1. Each factor in front of the exponent changes parity so that resulting parity is negative. Therefore the possible values of l can be only $l = 1$ or $l = 3$. The factor z keeps the projection $m = 0$ (no ϕ -dependence) but the product xy can be presented as

$$xy = \frac{1}{4i}[(x + iy)]^2 - (x - iy)^2] = \frac{r^2 \sin^2 \theta}{4i}(e^{2i\phi} - e^{-2i\phi}), \quad (3)$$

Therefore the function contains with equal probabilities $m = 2$ and $m = -2$. Then only $l = 3$ is possible.

PROBLEM 2. *a.* Since the $(2l + 1)$ states of the same l -multiplet are degenerate, the full degeneracy is a sum of such odd numbers. $d = 2$ would mean that two s states are degenerate which is impossible. $d = 7$ always occurs for a multiplet of f states ($l = 3$). $d = 9$ can happen in the case of a g -level ($l = 4$), or degenerate s, p and d states as in the shell $n = 2$ of the hydrogen atom.

b. The ground state always has $l = 0$, the first excited state can be either again $l = 0$, dimension $d = 1$, or $l = 1$, dimension $d = 3$; they also can be degenerate as in the hydrogen atom, $d = 4$. Other values of d are impossible.

PROBLEM 3. *a.* The low energy limit means $kR \ll 1$ where

$$k = \sqrt{\frac{2mE}{\hbar^2}}. \quad (4)$$

Then we can consider the s -wave only. In the outside region the wave function is $\psi(r) = u(r)/r$, where

$$u(r) = B \sin(kr + \delta), \quad r > R, \quad (5)$$

with the unknown phase δ . Under the barrier

$$u(r) = A \sinh(\kappa r), \quad \kappa = \sqrt{\frac{2m(U_0 - \epsilon)}{\hbar^2}}, \quad (6)$$

where, in the low energy case, $E < U_0$. Matching the logarithmic derivative at $r = R$, we obtain

$$\tan(kR + \delta) = k \frac{\tanh(\kappa R)}{\kappa}. \quad (7)$$

At low energy the s -wave phase δ is a linear function of k ; then we can substitute $\tan(kR + \delta) \approx kR + \delta$ and neglect E as compared to U_0 in the definition (6) of κ which leads to

$$\delta = kR \left(\frac{\tanh(\kappa_0 R)}{\kappa_0 R} - 1 \right), \quad \kappa_0 = \sqrt{\frac{2mU_0}{\hbar^2}}. \quad (8)$$

Indeed, the phase $\delta \propto k$. In the same approximation we find the cross section

$$\sigma = \frac{4\pi}{k^2} \sin^2 \delta \approx \frac{4\pi}{k^2} \delta^2 = 4\pi R^2 \left(\frac{\tanh(\kappa_0 R)}{\kappa_0 R} - 1 \right)^2. \quad (9)$$

b. In the limit of $U_0 \rightarrow \infty$, $\kappa_0 \rightarrow \infty$,

$$\tanh(\kappa_0 R) \rightarrow 1, \quad \frac{\tanh(\kappa_0 R)}{\kappa_0 R} \rightarrow 0, \quad \sigma \rightarrow 4\pi R^2, \quad (10)$$

four times larger than the classical value. The scattering phase is in this limit

$$\delta(k) = -kR, \quad (11)$$

in agreement with the physical meaning of the reflection from the impenetrable wall which reduces the phase of the wave by the path inside the well.

c. In the Born approximation the scattering amplitude is

$$f(q) = -\frac{m}{2\pi\hbar^2} \int d^3r e^{i(\mathbf{q}\cdot\mathbf{r})} U(r). \quad (12)$$

The simple integration gives ($\xi = qR$)

$$f(q) = -\frac{2mU_0R^3(\sin \xi - \xi \cos \xi)}{\hbar^2 \xi^3}. \quad (13)$$

The low energy case corresponds to $\xi \ll 1$. Then the scattering is isotropic,

$$f(q) \approx -\frac{2mU_0R^3}{3\hbar^2}, \quad (14)$$

and the cross section is

$$\sigma = 4\pi f^2 = \frac{16\pi m^2 U_0^2 R^6}{9\hbar^4}. \quad (15)$$

The same result follows from the exact s -wave expression (9) if the potential is weak, $\kappa_0 R \ll 1$: then

$$\tan(\kappa_0 R) \approx \kappa_0 R \left[1 - \frac{(\kappa_0 R)^2}{3} \right], \quad (16)$$

$$\sigma \approx 4\pi R^2 \frac{(\kappa_0 R)^4}{9}, \quad (17)$$

which coincides with (15).