PROBLEM 1.  a. /3/ A wave function of a particle is
\[ \psi(r) = A(x^2 - y^2)ze^{-\beta r}. \]  
Determine the probabilities of various values of \( l, m \), and parity in this state.

b. /5/ The same for the function
\[ \psi(r) = A(x^2 + y^2)ze^{-\beta r}. \]  
Here it may be useful to use the Legendre polynomial
\[ P_3(\cos \theta) = \frac{5}{2} \cos^3 \theta - \frac{3}{2} \cos \theta. \]  

PROBLEM 2. /6/ For a spinless particle in a three-dimensional isotropic potential \( U(r) \) (of arbitrary shape),

a. is it possible to have a level with the degeneracies \( d = 2, d = 7, d = 9 \)?

b. what are the possible degeneracies of the first excited level?

For each case give an example.

PROBLEM 3. /16/ a. Determine the scattering amplitude and cross section for a low energy particle (formulate explicitly what is the low energy approximation and when it can be valid) from a repulsive potential
\[ U(r) = \begin{cases} U_0 > 0, & r < R, \\ 0, & r > R. \end{cases} \]  

b. Check your answer in the limit \( U_0 \rightarrow \infty \).

c. Compare the result (a) with the Born approximation. What is the condition of validity for the Born approximation?
SOLUTION, Problem 1.

a. This case is quite similar to that in the main Midterm. In terms of raising, \( r_+ \), and lowering, \( r_- \), component of the vector \( \mathbf{r} \), the function (1), can be expressed as

\[
\psi(\mathbf{r}) = \frac{A}{2}(r_+^2 + r_-^2)ze^{-\beta r} \propto (Y_{22} + Y_{2-2})Y_{10}.
\]

(5)

Therefore we uniquely determine negative parity and \( l = 3 \) since another possibility with negative parity, \( l = 1 \), is incompatible with the projections \( m = \pm 2 \) which appear equiprobably.

b. This case is a little more complicated. It is convenient to rewrite the function separating the spherically invariant part \( r^2 \),

\[
\psi(\mathbf{r}) = A(r^2 - z^2)ze^{-\beta r} \propto \eta - \eta^3, \quad \eta = \cos \theta.
\]

(6)

Here the projection \( m = 0 \) is defined uniquely and parity is still negative. However this function is a superposition of \( Y_{10} \) and \( Y_{30} \), or, in terms of the Legendre polynomials \( P_l(\eta) \),

\[
\eta - \eta^3 = aP_1 + bP_3 = a(P_1 - P_3),
\]

(7)

where the last equality follows from the fact that in the forward direction (\( \eta = 1 \)) all \( P_1 = 1 \), and the left hand side vanishes, so that \( a + b = 0 \). Using the relation between Legendre polynomials and spherical functions, we obtain

\[
\eta - \eta^3 = a\sqrt{4\pi} \left( \frac{Y_{10}}{3} - \frac{Y_{30}}{7} \right).
\]

(8)

Since the spherical function are orthonormalized, the needed probabilities \( w_l \) of \( l = 3 \) and \( l = 1 \) are, respectively,

\[
w_1 = \frac{7}{10}, \quad w_3 = \frac{3}{10}.
\]

(9)