PRACTICE PROBLEMS

1. Find the commutators

$$[\hat{l}_i, \hat{\mathbf{p}}^2]; \quad [\hat{l}_i, (\hat{\mathbf{p}} \cdot \hat{\mathbf{r}})^2]; \quad [\hat{l}_i, (\hat{\mathbf{p}} \cdot \hat{\mathbf{r}})\mathbf{r}]; \quad [\hat{l}_i, \hat{a}_k^\dagger]; \quad [\hat{l}_i, \hat{x}_j \hat{p}_k]. \tag{1}$$

   Here $\hat{l}_i$ is the orbital momentum component and $\hat{a}_k^\dagger$ is the creation operator for the oscillator quanta along the axis k.

2. Find the commutator $[\hat{l}_i, \hat{l}'_j]$ where $\hat{\mathbf{l}}$ and $\hat{\mathbf{l}}'$ are the orbital momentum operators with respect to the origin and the point $\mathbf{a}$, respectively.

3. Decompose the orbital momentum of a two-body system into the relative motion orbital momentum and that of the center-of-mass motion.

4. Write down a normalized wave function with orbital momentum l and its z-projection m for a particle bound in a very thin spherical shell at a distance $r = R$ from the center.

5. Write down the orbital momentum operator in the momentum representation and the angular part of the momentum representation wave function for a particle in a state with orbital momentum quantum numbers l and m.

6. The wave function of a particle is

$$\psi(\mathbf{r}) = Axye^{-r^2/a^2}. \tag{2}$$

   Find the probabilities of various values of l, m and parity.

7. For a particle with orbital momentum l find the projection operator $\hat{P}(m)$ which selects the state with a certain value m of $\hat{l}_z$.

8. *a.* For a particle with orbital momentum $l = 1$ construct the wave function $\psi(\theta, \phi)$ for the state with the zero projection of the orbital momentum vector onto the axis defined by the polar angle α and asimuthal angle β.

   b. The same for the state with projection $l_x = 1$ onto the x-axis.

9. For a particle in the state with the orbital momentum quantum numbers $l = 1$ and m find

   a. the probabilities of various values m' of the orbital momentum projection on the axis z' which has an angle α with the respect to the z-axis;

   b. the expectation values $\langle \hat{l}_x^3 \rangle$ and $\langle \hat{l}_y^3 \rangle$.

10. A particle is placed into a two-dimensional isotropic ($\omega_x = \omega_y$) harmonic oscillator field.

    a. Find the energy levels and the wave functions of the stationary states.

    b. Determine the degeneracy of the stationary states.

    c. For the state with the Cartesian quantum numbers $n_x = n_y = 1$ find the probabilities of various values of the orbital momentum projection l_z.

    d. Determine the approximate value of the ground state energy using the trial function $\psi(\rho) = A\exp(-\beta\rho)$ and compare the result with the exact value.

11. For a spinless particle in a three-dimensional isotropic potential $U(r)$, is it possible to have a level with the degeneracies $d = 3, d = 4$?

12. Which l values are possible for the states of the three-dimensional isotropic harmonic oscillator with $N = 2$? Find the combination of the states which corresponds to the s-wave.

13. Consider the set of operators $a\hat{x}_i\hat{x}_j + b\hat{p}_i\hat{p}_j$ with some constants a and b. Show that, for a specific choice of those constants, these operators are constants of motion for a particle in the three-dimensional isotropic harmonic oscillator field; explain the underlying physics.

14. Find the expectation value $\langle r^s \rangle$ for the ground state of the hydrogen-like ion with the charge Z of the nucleus.

15. Find the average electric field created at large distances by the hydrogen atom in the $2p$-state with the projection $l_z = m$.

16. Determine the approximate value of the ground state energy for the hydrogen atom using the trial wave function $\psi(r) = A\exp(-\beta^2 r^2)$.

17. For a particle in a ground state of the spherically symmetric potential well,
$$U(r) = \begin{cases} 0, & r \le R, \\ \infty, & r > R, \end{cases} \tag{3}$$
    find the momentum distribution.

18. Consider a rotor, a particle with the rotational Hamiltonian
$$\hat{H} = \frac{\hbar^2 \hat{\mathbf{l}}^2}{2I}, \tag{4}$$
    where I is the moment of inertia. At the initial moment the wave function is given by
$$\Psi(t = 0) = A\cos^2\theta. \tag{5}$$
    Find the wave function as a function of time.

19. *a.* Using the Born approximation, find the scattering amplitude, differential and total cross section for the Yukawa potential

$$U(r) = \frac{g}{r} e^{-\mu r}. \tag{6}$$

b. Investigate the limiting cases of long and short wavelengths and establish the criteria of validity of the Born approximation.

20. Calculate in the Born approximation the differential and total cross section of scattering of electrons by the hydrogen atom in the ground state.

21. A particle is scattered on a molecule that has a form of a plane square with a side length a and four identical atoms in the corners; the interaction potential between the particle and each atom is $U(\mathbf{r})$. Find the relation between the scattering amplitudes, differential and total cross sections for the scattering off the molecule and scattering off an individual atom (consider the cases of low and high energy).

22. At energy $E \to 0$ compare the exact scattering amplitude for a repulsive potential $U(r) \geq 0$ to the Born amplitude for the same potential. Show that the Born cross section is larger than the exact one.

23. *a.* Find the s-wave phase shift for the scattering off the attractive well of radius R and depth U_0. Calculate the cross section at low energy.

    b. Show that for the shallow well the result coincides with that in the Born approximation.

    c. Show that as the depth of the well increases and reaches the critical value for the appearance of the bound state, the cross section reveals the resonance.

24. Taking into account the s- and p-waves, find the angular distribution of the scattered particles and the total cross section.