PHY-852 QUANTUM MECHANICS II
Homework 5, 30 points
February 13 - 20, 2002
Hydrogen atom, spherical well and harmonic oscillator.
Reading: *Merzbacher*, Chapters 11 and 12.

1. /6/ Consider a particle of mass m in the potential $U(r) = -U_0$ for $r < a$ and $U = 0$ for $r > a$. Find the values of the depth U_0 corresponding to the appearance in the well of new levels with orbital momentum $l = 1$.

2. /12/ *a.* Show that the Hamiltonian of a particle of mass m in a spherically symmetric potential $U(r)$ can be presented in the form

$$\hat{H} = \frac{\hat{p}_r^2}{2m} + \frac{\hbar^2 \hat{\mathbf{l}}^2}{2mr^2} + U(r) \tag{1}$$

where the radial momentum $\hat{p}_r = -i\hbar(1/r)(\partial/\partial r)r$ is canonically conjugate to the radial coordinate $\hat{r}$. Derive the Heisenberg operator equations of motion for $\hat{r}$ and $\hat{p}_r$.

b. Show that for an operator $\hat{O}$ without explicit time dependence the expectation value $\langle d\hat{O}/dt\rangle = 0$ in any state of the discrete spectrum. Using $\hat{O} = \hat{r}^q$ derive the relation

$$\langle p_r r^{q-1}\rangle = -\frac{i\hbar}{2}(q-1)\langle r^{q-2}\rangle. \tag{2}$$

c. Taking $\hat{O} = \hat{p}_r \hat{r}^{q+1}$ and eliminating $\hat{p}_r^2$ and $d\hat{p}_r/dt$ with the help of the Hamiltonian and equations of motion for $\hat{p}_r$, obtain the recurrent relation for the Coulomb potential $U(r) = Ze^2/r$ and stationary state with quantum numbers n and l

$$2E_{nl}\langle r^q\rangle_{nl} + Ze^2(2q+1)\langle r^{q-1}\rangle_{nl} + \frac{q\hbar^2}{m}\left[\frac{q^2-1}{4} - l(l+1)\right]\langle r^{q-2}\rangle_{nl} = 0. \tag{3}$$

d. Taking in (3) suitable values of the parameter q, find the virial theorem and calculate the expectation values $\langle 1/r\rangle_{nl}$, $\langle r\rangle_{nl}$ and $\langle r^2\rangle_{nl}$; express the result in the units related to the Bohr radius.

e. For the isotropic harmonic oscillator, $U(r) = (1/2)m\omega^2 r^2$, obtain the recurrent relation analogous to eq. (3), recover the virial theorem and calculate $\langle r^2\rangle_{Nl}$ and $\langle r^4\rangle_{Nl}$ for the stationary state with main quantum number N and orbital momentum l.

3. /12/ For the hydrogen atom in the ground state find the expectation values of the electrostatic potential $\langle\varphi(\mathbf{R})\rangle$, the electric field $\langle\vec{\mathcal{E}}(\mathbf{R})\rangle$, and the product of the components $\langle\mathcal{E}_i(\mathbf{R})\mathcal{E}_k(\mathbf{R})\rangle$ at an arbitrary point $\mathbf{R}$; explain the asymptotic, $R \to \infty$, behavior.