NAME..

PHY-852: QUANTUM MECHANICS I
Quiz 2
February 6, 2002

PROBLEM. For a particle in a central field in the state with the orbital momentum l and its projection $l_z = m$, calculate the expectation value $\langle l_{\mathbf{n}} \rangle$ and the mean square fluctuation

$$\langle (\Delta l_{\mathbf{n}})^2 \rangle = \langle l_{\mathbf{n}}^2 \rangle - \langle l_{\mathbf{n}} \rangle^2 \tag{1}$$

where $l_{\mathbf{n}}$ is the orbital momentum projection onto the axis $\mathbf{n}$ with the azimuthal angle α and polar angle β. Explain what happens at $\beta = 0$.

SOLUTION.

The operator under question is

$$l_{\mathbf{n}} = (\mathbf{l} \cdot \mathbf{n}) = l_x n_x + l_y n_y + l_z n_z = l_x \sin\beta\cos\alpha + l_y \sin\beta\sin\alpha + l_z\cos\beta. \quad (2)$$

Its expectation value is

$$\langle l_{\mathbf{n}} \rangle = \langle l_z \rangle \cos\beta = m\cos\beta. \quad (3)$$

In calculating $\langle l_{\mathbf{n}}^2 \rangle$, the operators $l_x l_z, l_z l_x$ and $l_y l_z, l_z l_y$ have no expectation values (total change of projection is ± 1); the operators $l_x l_y$ and $l_y l_x$ cancel each other, see *Problem 1b*, Homework 2. It results in

$$\langle l_{\mathbf{n}}^2 \rangle = \langle l_x^2 \rangle \sin^2\beta\cos^2\alpha + \langle l_y^2 \rangle \sin^2\beta\sin^2\alpha + \langle l_z^2 \rangle \cos^2\beta, \quad (4)$$

or using the results of the same *Problem 1*,

$$\langle l_{\mathbf{n}}^2 \rangle = \frac{1}{2}[l(l+1) - m^2]\sin^2\beta + m^2\cos^2\beta. \quad (5)$$

The mean square fluctuation is

$$\langle (\Delta l_{\mathbf{n}})^2 \rangle = \frac{1}{2}\sin^2\beta[l(l+1) - m^2]. \quad (6)$$

It vanishes at $\beta = 0$ when the direction of $\mathbf{n}$ coincides with the polar axis (then $l_{\mathbf{n}} = l_z$ has a certain value).