PROBLEM. For a particle in a central field in the state with the orbital momentum \( l \) and its projection \( l_z = m \), calculate the expectation value \( \langle l_n \rangle \) and the mean square fluctuation

\[
(\langle \Delta l_n \rangle^2) = (l_n^2) - (l_n)^2
\]  

(1)

where \( l_n \) is the orbital momentum projection onto the axis \( n \) with the azimuthal angle \( \alpha \) and polar angle \( \beta \). Explain what happens at \( \beta = 0 \).
SOLUTION.

The operator under question is

\[ l_n = (1 \cdot n) = n_x l_x + n_y l_y + n_z l_z = l_x \sin \beta \cos \alpha + l_y \sin \beta \sin \alpha + l_z \cos \beta. \quad (2) \]

Its expectation value is

\[ \langle l_n \rangle = \langle l_z \rangle \cos \beta = m \cos \beta. \quad (3) \]

In calculating \( \langle l_n^2 \rangle \), the operators \( l_x l_z, l_z l_x \) and \( l_y l_z, l_z l_y \) have no expectation values (total change of projection is \( \pm 1 \)); the operators \( l_z l_y \) and \( l_y l_z \) cancel each other, see Problem 1b, Homework 2. It results in

\[ \langle l_n^2 \rangle = \langle l_x^2 \rangle \sin^2 \beta \cos^2 \alpha + \langle l_y^2 \rangle \sin^2 \beta \sin^2 \alpha + \langle l_z^2 \rangle \cos^2 \beta, \quad (4) \]

or using the results of the same Problem 1,

\[ \langle l_n^2 \rangle = \frac{1}{2} [l(l+1) - m^2] \sin^2 \beta + m^2 \cos^2 \beta. \quad (5) \]

The mean square fluctuation is

\[ \langle (\Delta l_n)^2 \rangle = \frac{1}{2} \sin^2 \beta [l(l+1) - m^2]. \quad (6) \]

It vanishes at \( \beta = 0 \) when the direction of \( n \) coincides with the polar axis (then \( l_n = l_z \) has a certain value).