NAME..

PHY-852: QUANTUM MECHANICS I
Quiz 3
February 2002

PROBLEM. A three-dimensional square potential well supports only one loosely bound level with the ratio of binding energy to the depth of the well $\epsilon/U_0 = 1/60$. Estimate the probability to find the particle outside the well.

SOLUTION.

The bound state has to have $l = 0$. The s-wave solution has a standard form $\psi(r) = u(r)/r$,

$$u(r) = \begin{cases} A\sin(kr), & k = \sqrt{2m(U_0 - \epsilon)/\hbar^2}, \quad 0 \le r \le R. \\ B\exp(-\kappa r), & \kappa = \sqrt{2m\epsilon/\hbar^2}, \quad r \ge R, \end{cases} \tag{1}$$

where R is the radius of the well. From the continuity requirements

$$A\sin(kR) = Be^{-\kappa R}, \quad kA\cos(kR) = -\kappa Be^{-\kappa R}, \tag{2}$$

and the equation for binding energy is

$$k\cot(kR) = -\kappa. \tag{3}$$

Since the level is very loosely bound,

$$kR \approx \frac{\pi}{2}, \quad k \approx \sqrt{\frac{2mU_0}{\hbar^2}}. \tag{4}$$

The internal probability

$$I_i = A^2 \int_0^R dr\, \sin^2(kr) = A^2 \left(\frac{R}{2} - \frac{\sin(2kR)}{4k} \right) \approx A^2 \frac{R}{2}; \tag{5}$$

the external probability

$$I_e = B2 \int_R^\infty dr\, e^{-2\kappa R} = B^2 \frac{\exp(-2\kappa R)}{2\kappa}, \tag{6}$$

or, using the first condition (2) and (4),

$$I_e = A^2 \frac{\sin^2(kR)}{2\kappa} \approx A^2 \frac{1}{2\kappa}. \tag{7}$$

Thus,

$$\frac{I_e}{I_i} = \frac{1}{\kappa R}. \tag{8}$$

But

$$\frac{\epsilon}{U_0} = \frac{(\kappa R)^2}{(kR)^2} \approx \frac{4}{\pi^2}(\kappa R)^2 = \frac{1}{60}, \tag{9}$$

which gives

$$(\kappa R)^2 \approx \frac{\pi^2}{240} \quad \rightsquigarrow \quad \frac{I_e}{I_i} \approx \frac{\sqrt{240}}{\pi} \approx 5. \tag{10}$$