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PHY-852: QUANTUM MECHANICS II

Quiz 5

April 22, 2002

PROBLEM. An atomic configuration is characterized by the total orbital momentum of electrons L and their total spin S.

a. Draw a scheme of the fine structure level splitting under the influence of the spin-orbit interaction $H_{ls} = A\mathbf{L}\cdot\mathbf{S}$. Indicate the number of components, their ordering and energy spacings between them.

b. Explain qualitatively what happens in the presence of the additional static magnetic field if the magnetic interaction is weaker than the spin-orbit interaction.

c. Explain qualitatively what happens in the presence of the additional static magnetic field if the magnetic interaction is stronger than the spin-orbit interaction.

SOLUTION.

a. With no spin-orbit coupling all $(2L+1)(2S+1)$ states with different combinations of the magnetic quantum numbers M_L and M_S are degenerate. Due to the spin-orbit coupling, these (LS) levels are split according to the value J of the total angular momentum $\mathbf{J} = \mathbf{L}+\mathbf{S}$; J is allowed by the triangle rule to change from $|L-S|$ to $L+S$. The number of split J-levels is equal to $(2J_{<}+1)$, where $J_{<}$ is the smallest of L and S. Each J-level is still $(2J+1)$-fold degenerate in the total projection M_J. It is easy to check that the total amount of states did not change,

$$\sum_{J=|L-S|}^{L+S} (2J+1) = (2L+1)(2S+1). \tag{1}$$

The displacement of a J-level from the original unperturbed position is given by the expectation value of the spin-orbit Hamiltonian H_{ls} for a given value of J (naively speaking, the angle between the $\mathbf{L}$ and $\mathbf{S}$):

$$E_J(L,S) = E^{\circ}(L,S) + \frac{A}{2}[J(J+1) - L(L+1) - S(S+1)]. \tag{2}$$

The spacing between two adjacent levels of the same configuration,

$$E_J(L,S) - E_{J-1}(L,S) = AJ, \tag{3}$$

is linear in J. In atoms the spin-orbit coupling prefers the levels with the minimum value of J to be lower, $A > 0$.

b. In the presence of the weak magnetic field B, the J-levels are weakly split in the projection $M_J \equiv M$ of the total angular momentum onto the direction of the field. The splitting (Zeeman effect) is fan-like,

$$\Delta E_{JM}(L,S) = -g(J,L,S)MB, \tag{4}$$

where the effective gyromagnetic ratio (Landé factor)

$$g(J, L, S) = \frac{g_l + g_s}{2} + \frac{g_l - g_s}{2J(J+1)}[L(L+1) - S(S+1)] \tag{5}$$

is expressed in terms of the orbital, g_l, and spin, g_s, gyromagnetic ratios,

$$g_l = \mu_B, \quad g_s = 2\mu_B, \quad \mu_B = \frac{e\hbar}{2m_e c}. \tag{6}$$

c. If the effects of the magnetic field are stronger than the fine structure splitting found in point (a), then we first have to account for the magnetic splitting due to the separate alignment of **L** and **S** along the magnetic field (the spin-orbit coupling is broken):

$$E_{M_L M_S} = -(g_l M_L + g_s M_S)B = -\mu_B(M_L + 2M_S)B. \tag{7}$$

The spin-orbit interaction in the first order leads only to the shift of each $(M_L M_S)$-level:

$$\Delta E_{M_L M_S} = \langle M_L M_S | H_{ls} | M_L M_S \rangle = A M_L M_S. \tag{8}$$

In all cases the constants of motion are the magnitudes L and S and the total projection $M = M_J = M_L + M_S$.