NAME..

PHY-852: QUANTUM MECHANICS I
Quiz 3
February 2002

PROBLEM. A three-dimensional square potential well supports only one loosely bound level with the ratio of binding energy to the depth of the well $\epsilon/U_0 = 1/60$. Estimate the probability to find the particle outside the well.

SOLUTION.

The bound state has to have $l=0$. The s-wave solution has a standard form $\psi(r)=u(r)/r$,

$$u(r)=\begin{cases} A\sin(kr), & k=\sqrt{2m(U_0-\epsilon)/\hbar^2}, \quad 0\le r\le R. \\ B\exp(-\kappa r), & \kappa=\sqrt{2m\epsilon/\hbar^2}, \quad r\ge R, \end{cases} \tag{1}$$

where R is the radius of the well. From the continuity requirements

$$A\sin(kR)=Be^{-\kappa R}, \quad kA\cos(kR)=-\kappa Be^{-\kappa R}, \tag{2}$$

and the equation for binding energy is

$$k\cot(kR)=-\kappa. \tag{3}$$

Since the level is very loosely bound,

$$kR\approx\frac{\pi}{2}, \quad k\approx\sqrt{\frac{2mU_0}{\hbar^2}}. \tag{4}$$

The internal probability

$$I_i=A^2\int_0^R dr\,\sin^2(kr)=A^2\left(\frac{R}{2}-\frac{\sin(2kR)}{4k}\right)\approx A^2\frac{R}{2}; \tag{5}$$

the external probability

$$I_e=B2\int_R^\infty dr\,e^{-2\kappa R}=B^2\frac{\exp(-2\kappa R)}{2\kappa}, \tag{6}$$

or, using the first condition (2) and (4),

$$I_e=A^2\frac{\sin^2(kR)}{2\kappa}\approx A^2\frac{1}{2\kappa}. \tag{7}$$

Thus,

$$\frac{I_e}{I_i}=\frac{1}{\kappa R}. \tag{8}$$

But

$$\frac{\epsilon}{U_0}=\frac{(\kappa R)^2}{(kR)^2}\approx\frac{4}{\pi^2}(\kappa R)^2=\frac{1}{60}, \tag{9}$$

which gives

$$(\kappa R)^2\approx\frac{\pi^2}{240} \quad \rightsquigarrow \quad \frac{I_e}{I_i}\approx\frac{\sqrt{240}}{\pi}\approx 5. \tag{10}$$