Happy Halloween

Chapter 33
LC Circuits

RC circuits (chapter 28)

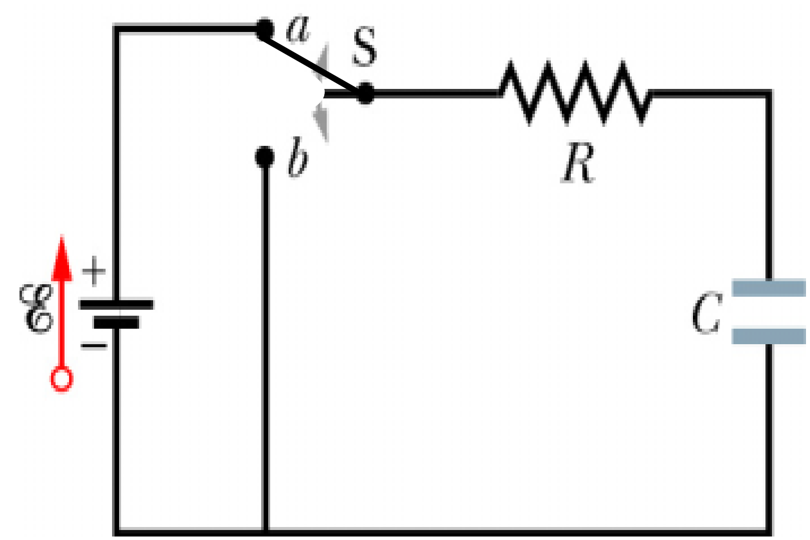
A resistor and capacitor in series

- Charging up a capacitor (switch to "a")

  The loop rule gives

$$\mathcal{E} - iR - \frac{q}{C} = 0$$

$$\mathcal{E} = \frac{dq}{dt}R + \frac{q}{C}$$



- The solution of this differential equation is:

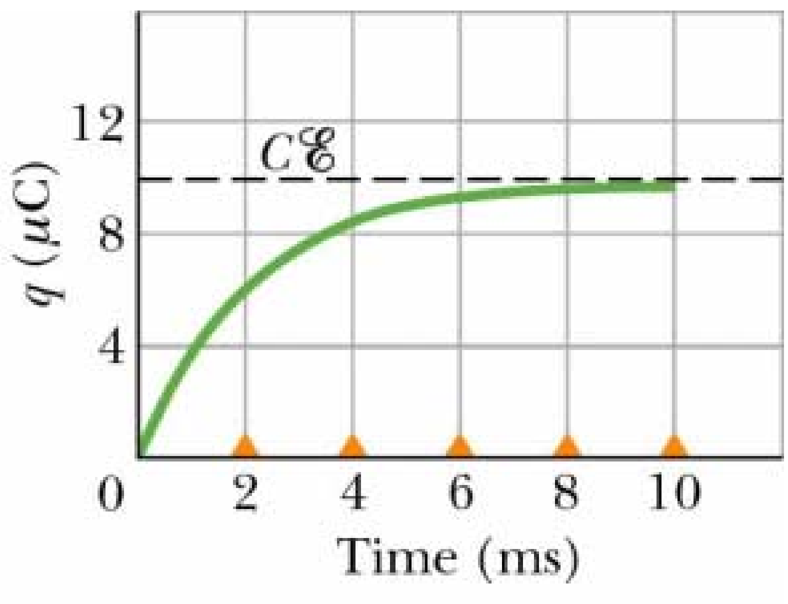
$$q = C\mathcal{E}\left(1 - e^{-t/\tau_c}\right)$$ where $$\tau_C = RC$$

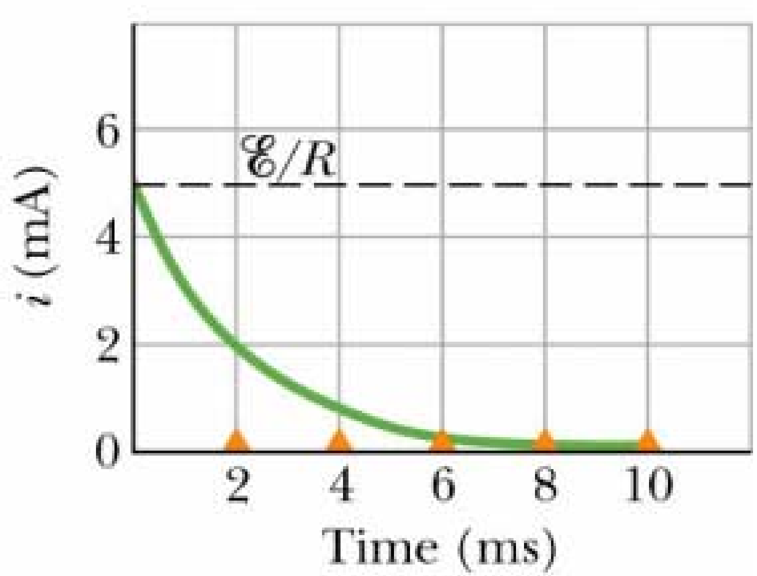
RC circuits (chapter 28)

- Graphical results

$$q = C\mathcal{E}(1 - e^{-t/\tau_c})$$

$$i = \frac{dq}{dt} = \frac{\varepsilon}{R} e^{-t/\tau_c}$$





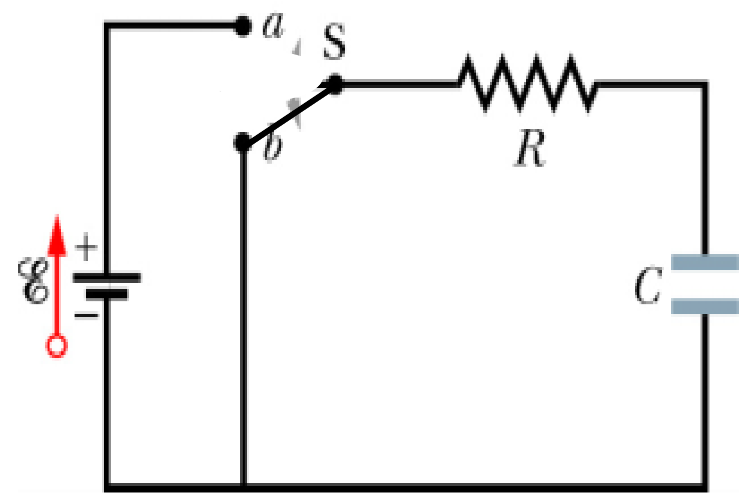
- Initially the current flows easily, but then drops exponentially as the capacitor get filled up.

RC circuits (chapter 28)

- Discharging capacitor (switch to "b") gives the differential equation:

$$iR + \frac{q}{C} = 0$$

$$\frac{dq}{dt}R + \frac{q}{C} = 0$$



- The solution is

$$q = q_0 e^{-t/\tau_c}$$

$$\tau_C = RC$$

Emf for an Inductor (chapter 31)

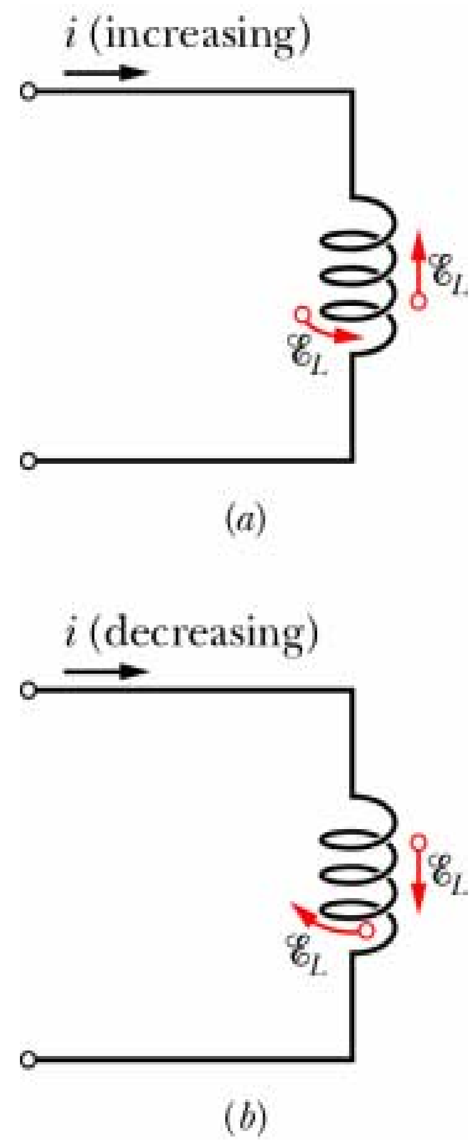
$$N\Phi_B = Li$$

- Self-inductance – changing current through an inductor gives an emf (voltage change) across the inductor denoted by

$$\mathcal{E}_L = -\frac{d(N\Phi_B)}{dt} = -\frac{d(Li)}{dt}$$

$$\mathcal{E}_L = -L\frac{di}{dt}$$

- Note the sign of this emf goes against the current change

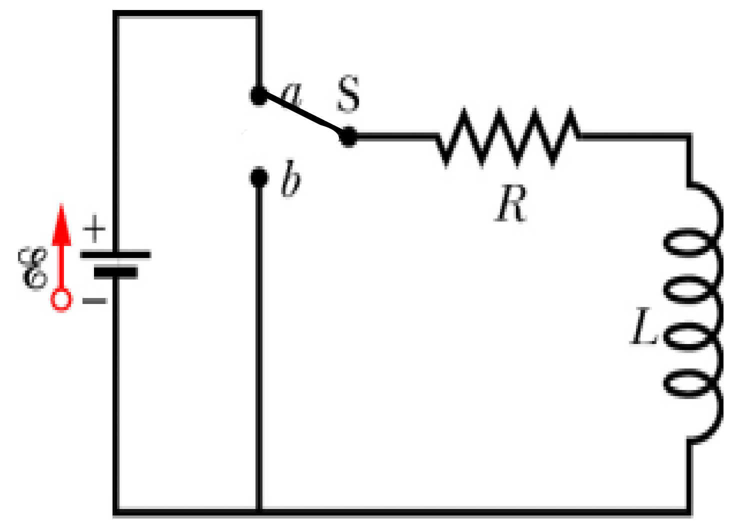


RL circuits (chapter 31 – Fig.17)

A resistor and inductor in series

- Charging up a inductor (switch to “a”)

  The loop rule gives



$$\mathcal{E} - iR - L\frac{di}{dt} = 0$$

- The solution of this differential equation is:

$$i = \frac{\mathcal{E}}{R}(1 - e^{-t/\tau_L})$$

Where $\tau_L = \frac{L}{R}$

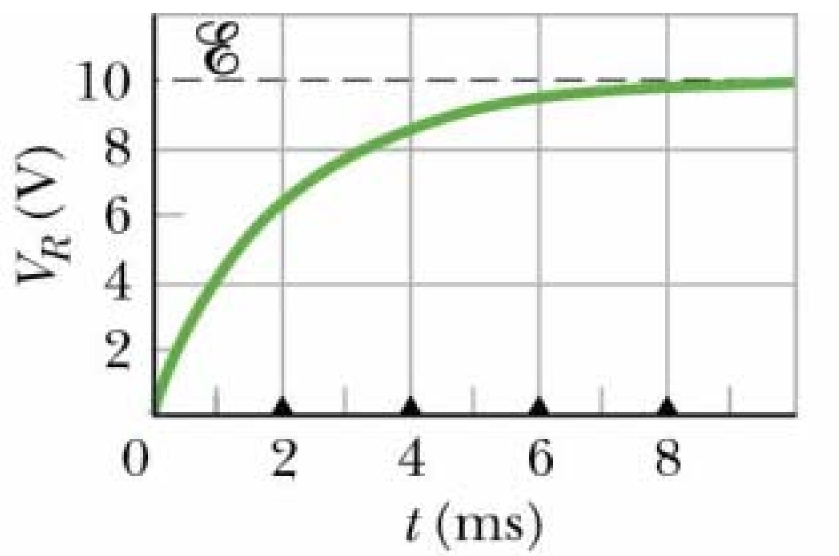
is the inductive time constant

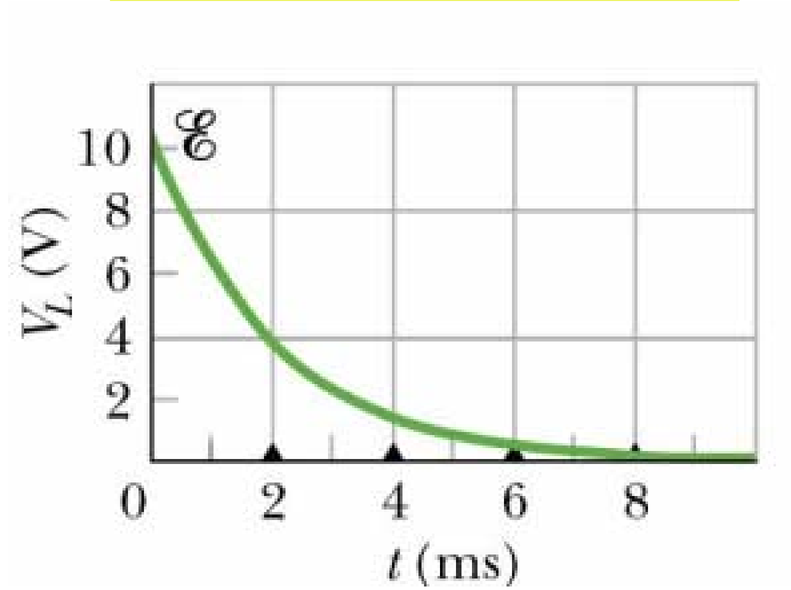
RL circuits (chapter 31 Fig. 19)

- Graphical results

$$V_R = iR = \mathcal{E}\,(1 - e^{-t/\tau_L})$$

$$V_L = -L\frac{di}{dt} = \mathcal{E}e^{-t/\tau_L}$$





- Initially inductor acts to oppose changes in current ($i=0$).
- Long time later, inductor acts like simple wire with ($i = \varepsilon/R$).

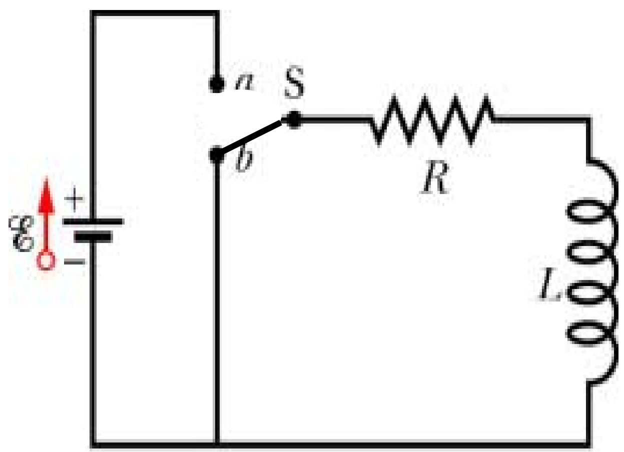
RL circuits (chapter 31)

- Circuit is opened (switch to "b")

$$-iR - L\frac{di}{dt} = 0$$

$$i = \frac{\varepsilon}{R}e^{-t/\tau_L} = i_0 e^{-t/\tau_L}$$

$$\tau_L = \frac{L}{R}$$



Energy stored in fields

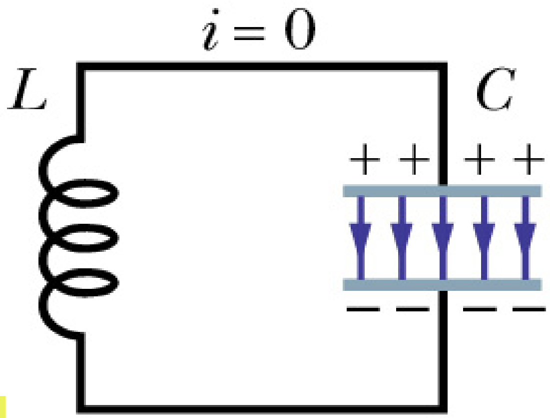
- Energy stored in the electric field of a capacitor (chapter 28)

$$U_E = \frac{1}{2}\frac{q^2}{C}$$

- Energy stored in the *B* field of an inductor (chapter 31)

$$U_B = \frac{1}{2} L i^2$$

LC Circuits



- LC Circuit – inductor & capacitor in series
- Find *q, i* and *V* vary sinusoidally with period *T* (angular frequency ω)

$$\omega = \frac{2\pi}{T}$$

- *E* field of capacitor and *B* field of inductor oscillate
- The energy oscillates between E field stored in the capacitor and the B field stored in the inductor

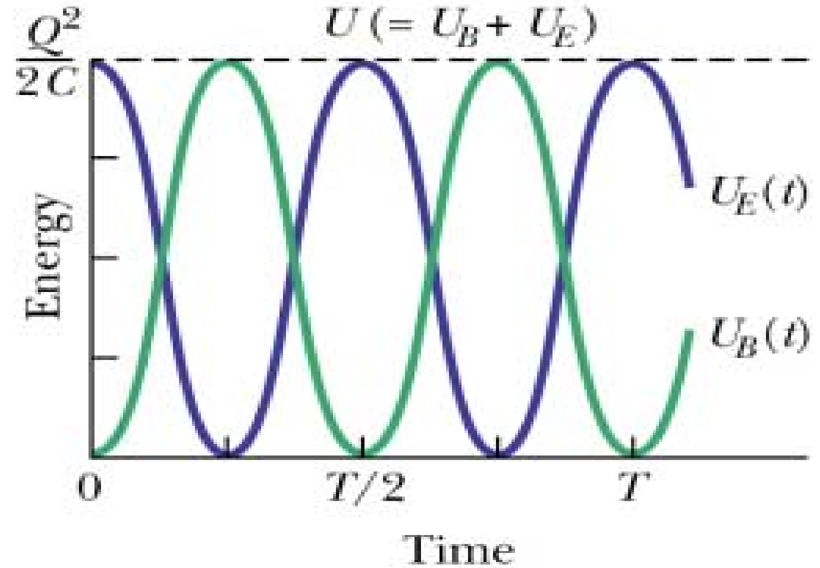
$$U_E = \frac{1}{2}\frac{q^2}{C}$$

$$U_B = \frac{1}{2}Li^2$$

LC Circuits

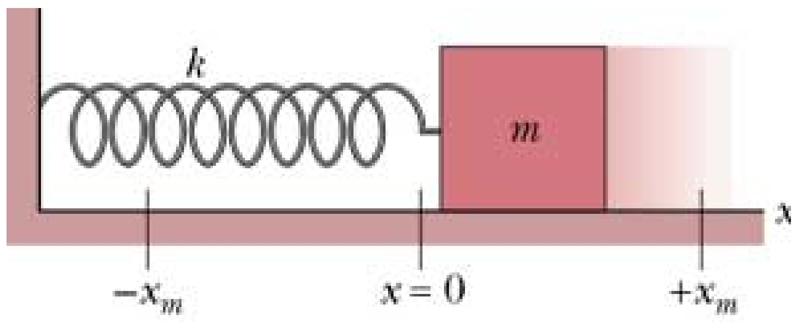
- Total energy of LC circuit

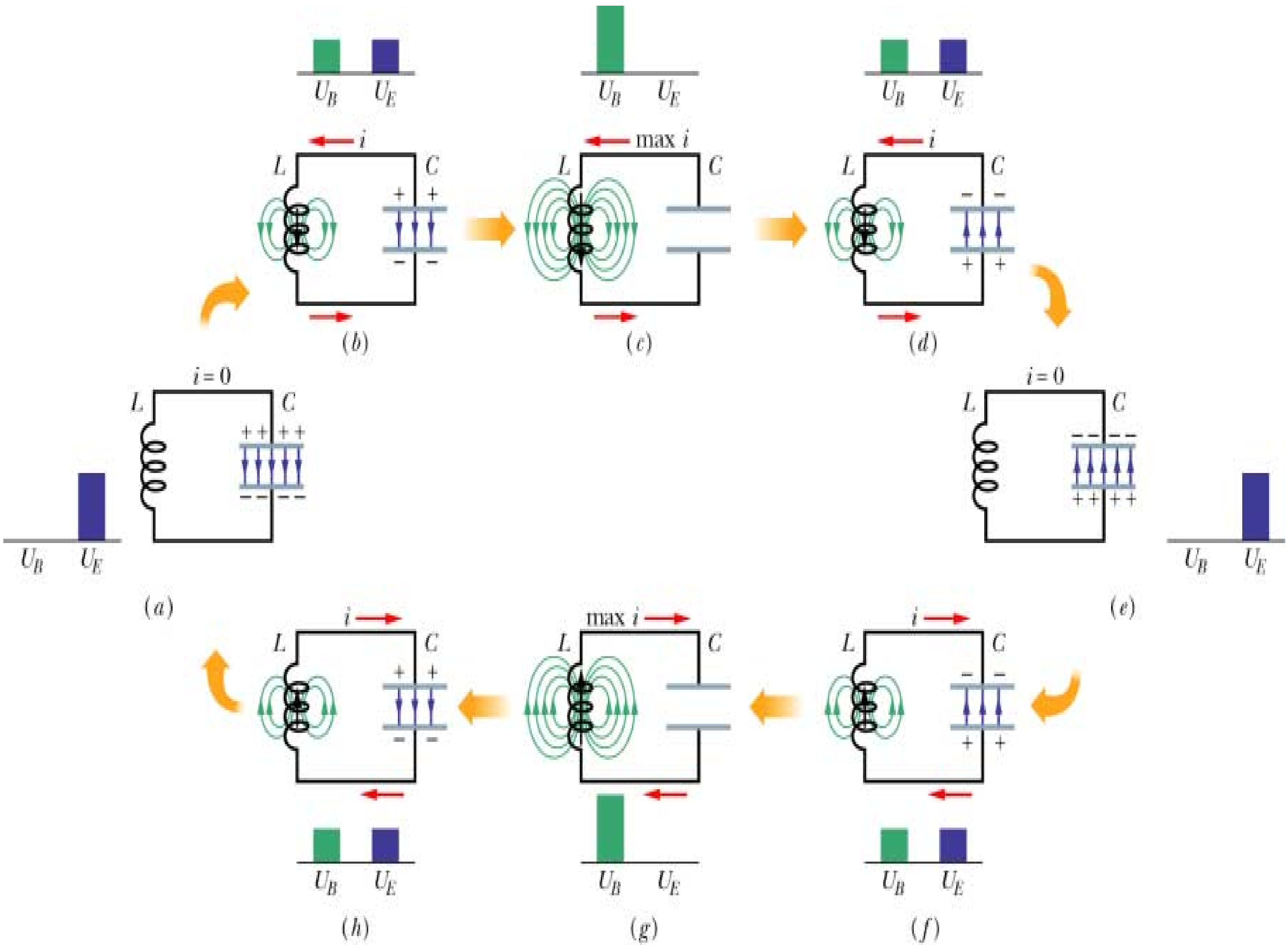
$$U = U_B + U_E = \frac{Li^2}{2} + \frac{q^2}{2C}$$



- Analogy to block-spring system (183)

$$U = \tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2$$





U_B
U_E
i
L
C
max i
(a)
(b)
(c)
(d)
(e)
(f)
(g)
(h)
i = 0

LC Circuits (checkpoint #1)

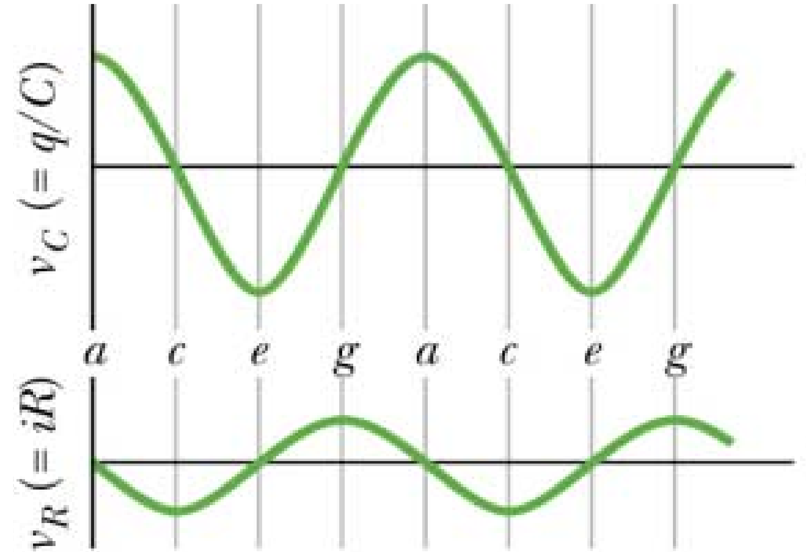
- A charged capacitor & inductor are connected in series at time t=0. In terms of period, T, how much later will the following reach their maximums:
  - *q* of capacitor

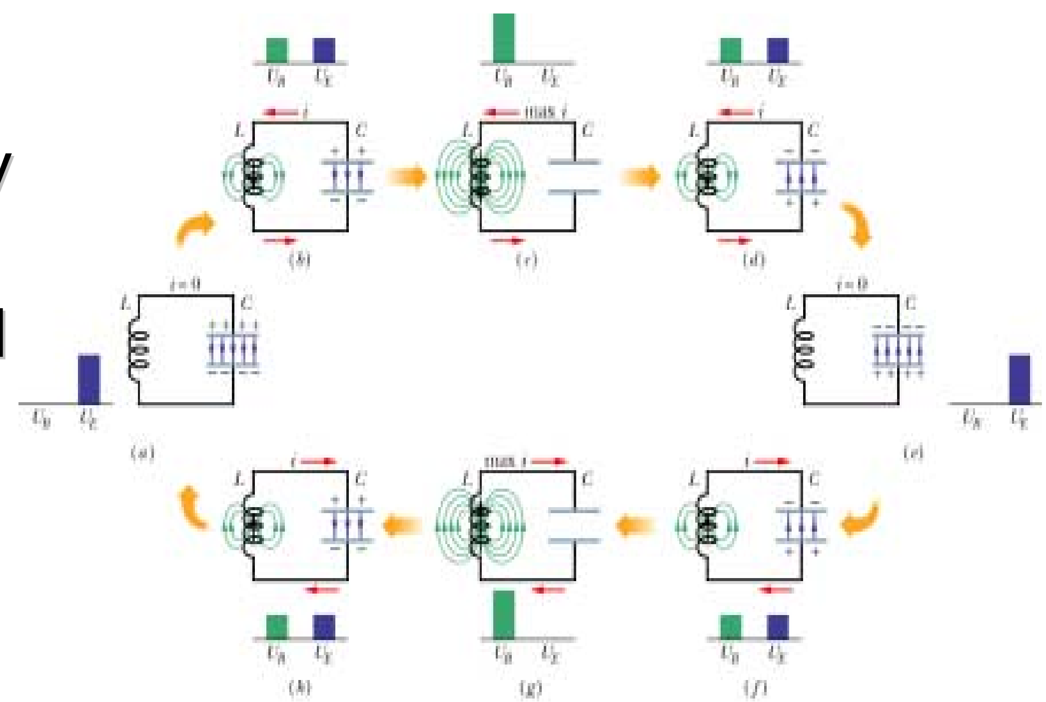
    T/2
  - V_c with original polarity

    T
  - Energy stored in *E* field

    T/2
  - The current

    T/4





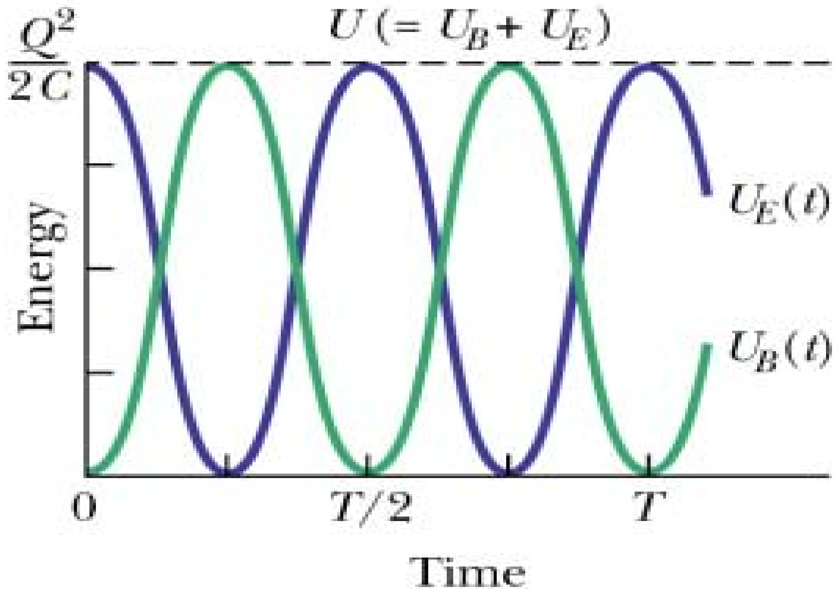
LC Circuits

- Total energy of LC circuit

$$U = U_B + U_E = \frac{Li^2}{2} + \frac{q^2}{2C}$$

- Total energy is constant

$$\frac{dU}{dt} = 0$$



- Differentiating gives

$$\frac{dU}{dt} = \frac{d}{dt}\left(\frac{Li^2}{2} + \frac{q^2}{2C}\right) = Li\frac{di}{dt} + \frac{q}{C}\frac{dq}{dt} = 0$$

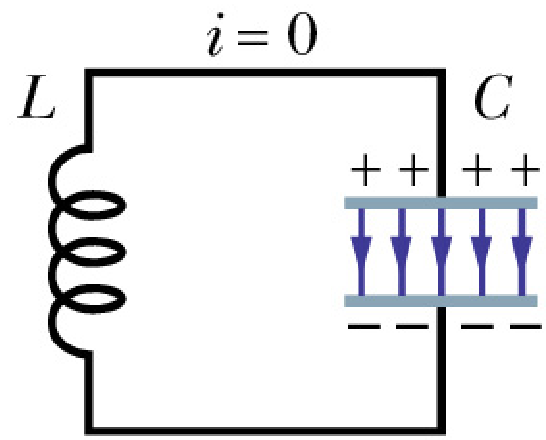
LC Circuits

- Using $$i = \frac{dq}{dt}$$ $$\frac{di}{dt} = \frac{d^2q}{dt^2}$$
- We obtain $$L\frac{d^2q}{dt^2} + \frac{1}{C}q = 0$$
- Solution is $$q = Q\cos(\omega t + \phi)$$
- where q is the variable, Q is the amplitude (maximum value for q)

LC Circuits

- The phase constant, ϕ, is determined by the conditions at time t=0 (or some other time)

$$q = Q\cos(\omega t + \phi)$$



- If $\phi = 0$ then at $t = 0$, $q = Q$

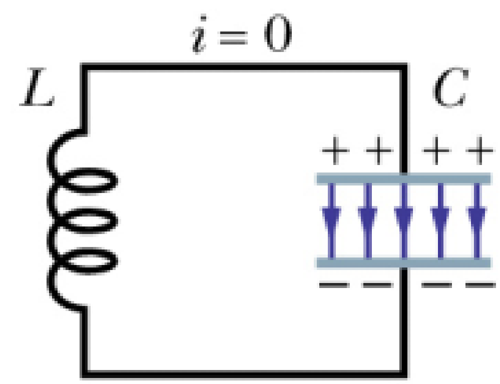
- I will take $\phi = 0$ for the rest of the lecture notes. To get the full result when you see ωt replace it by

$$\omega t + \phi$$

LC Circuits

- Charge of LC circuit

$$q = Q\cos(\omega t)$$



- Find current by $i = \frac{dq}{dt}$

$$i = \frac{d}{dt}\left[Q\cos(\omega t)\right] = -Q\omega\sin(\omega t)$$

- Amplitude I is

$$I = \omega Q$$

$$i = -I\sin(\omega t)$$

LC Circuits

- What is ω for an LC circuit?

$$q = Q\cos(\omega t)$$

$$\frac{d^2q}{dt^2} = -\omega^2 Q\cos(\omega t)$$

- Substitute into

$$L\frac{d^2q}{dt^2} + \frac{1}{C}q = 0$$

$$-L\omega^2 Q\cos(\omega t) + \frac{1}{C}Q\cos(\omega t) = 0$$

- Find ω for LC circuit is

$$\omega = \sqrt{\frac{1}{LC}}$$

LC Circuits

- The energy stored in an LC circuit at any time, *t*

$$U = U_B + U_E$$

- Substitute

$$q = Q\cos(\omega t)$$

$$i = -I\sin(\omega t)$$

$$U_E = \frac{q^2}{2C} = \frac{Q^2}{2C}\cos^2(\omega t)$$

$$U_B = \frac{Li^2}{2} = \frac{L}{2}\omega^2 Q^2 \sin^2(\omega t)$$

- Using

$$\omega = \sqrt{\frac{1}{LC}}$$

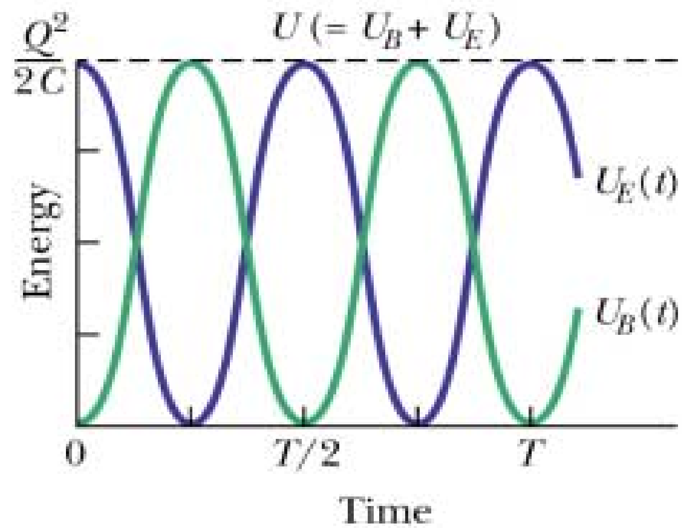
$$U_B = \frac{Q^2}{2C}\sin^2(\omega t)$$

LC Circuits

- Thus

$$U_E = \frac{Q^2}{2C}\cos^2(\omega t)$$

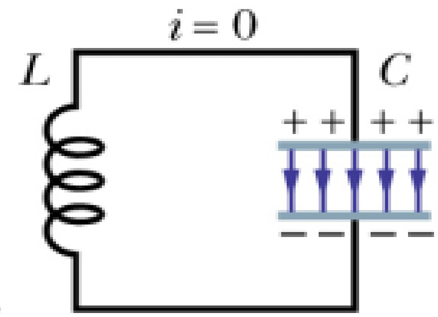
$$U_B = \frac{Q^2}{2C}\sin^2(\omega t)$$



- Maximum value for both $U_{E,\max} = U_{B,\max} = Q^2/2C$
- At any instant, sum is $U = U_B + U_E = Q^2/2C$
- When U_E = max, U_B = 0, and conversely, when U_B = max, U_E = 0

LC Circuits (checkpoint #2)-quiz

- Capacitor in LC circuit has $V_{C,max} = 15\ V$ and $U_{E,max} = 150\ J$. When capacitor has $V_C = 5\ V$ and $U_E = 50\ J$, what are the
  - 1) emf across the inductor?
  - 2) the energy stored in the *B* field?



- Apply the loop rule
  - Net potential difference around the circuit must be zero

$$V_L(t) = V_C(t)$$

$$U_{E,\max} = U_E(t) + U_B(t)$$

	a,	b,	c,	d,	e
1)	0,	5,	10,	15,	20 V
2)	0,	50,	100,	150,	200 J