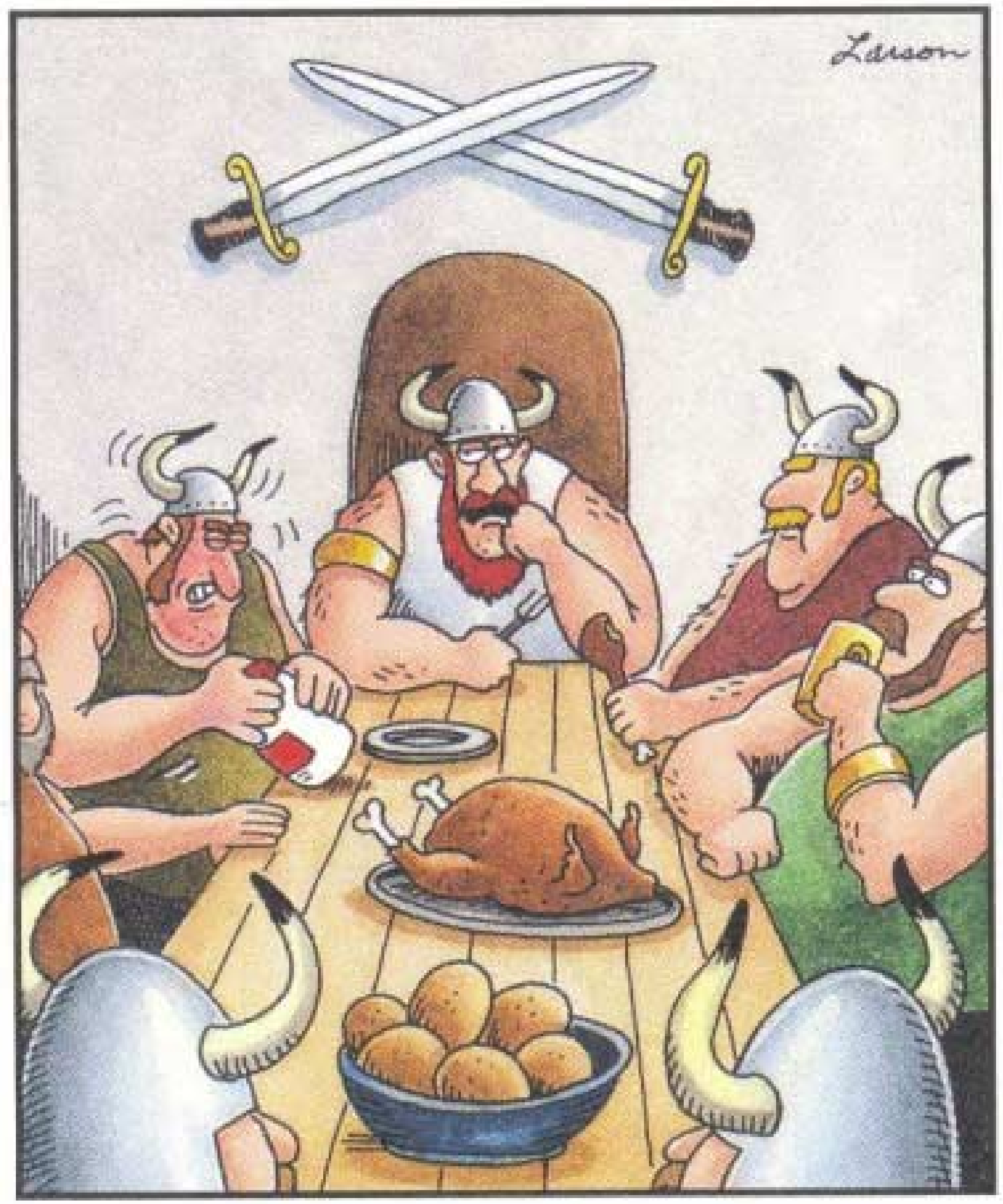
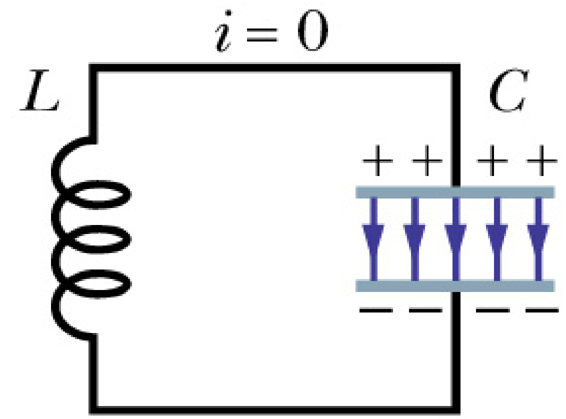
November 3rd

Chapter 33
RLC Circuits



"I can't believe this! ... Can't *anyone* here get the lid off the mayonnaise?"

LC Circuits



- LC Circuit – inductor & capacitor in series

- Find *q*, *i* and *V* vary sinusoidally with period *T* (angular frequency ω)

- The energy oscillates between E field stored in the capacitor and the B field stored in the inductor

$$\omega = \frac{2\pi}{T} = 2\pi f$$

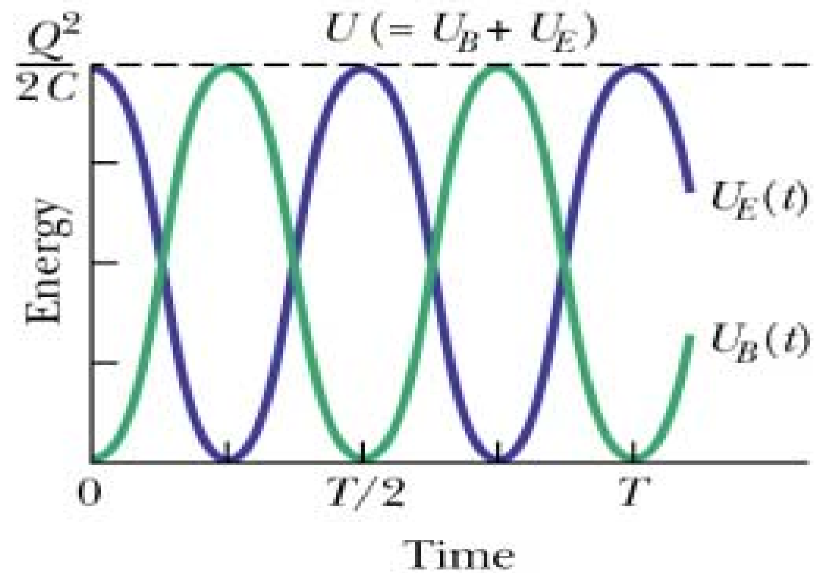
$$U_E = \frac{1}{2}\frac{q^2}{C}$$

$$U_B = \frac{1}{2}Li^2$$

LC Circuits

- Total energy of LC circuit

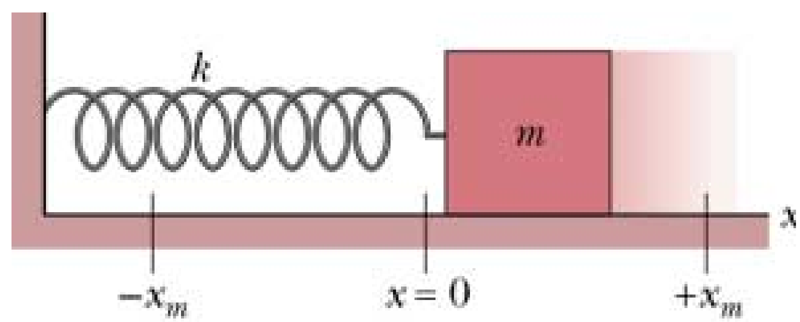
$$U = U_B + U_E = \frac{Li^2}{2} + \frac{q^2}{2C}$$

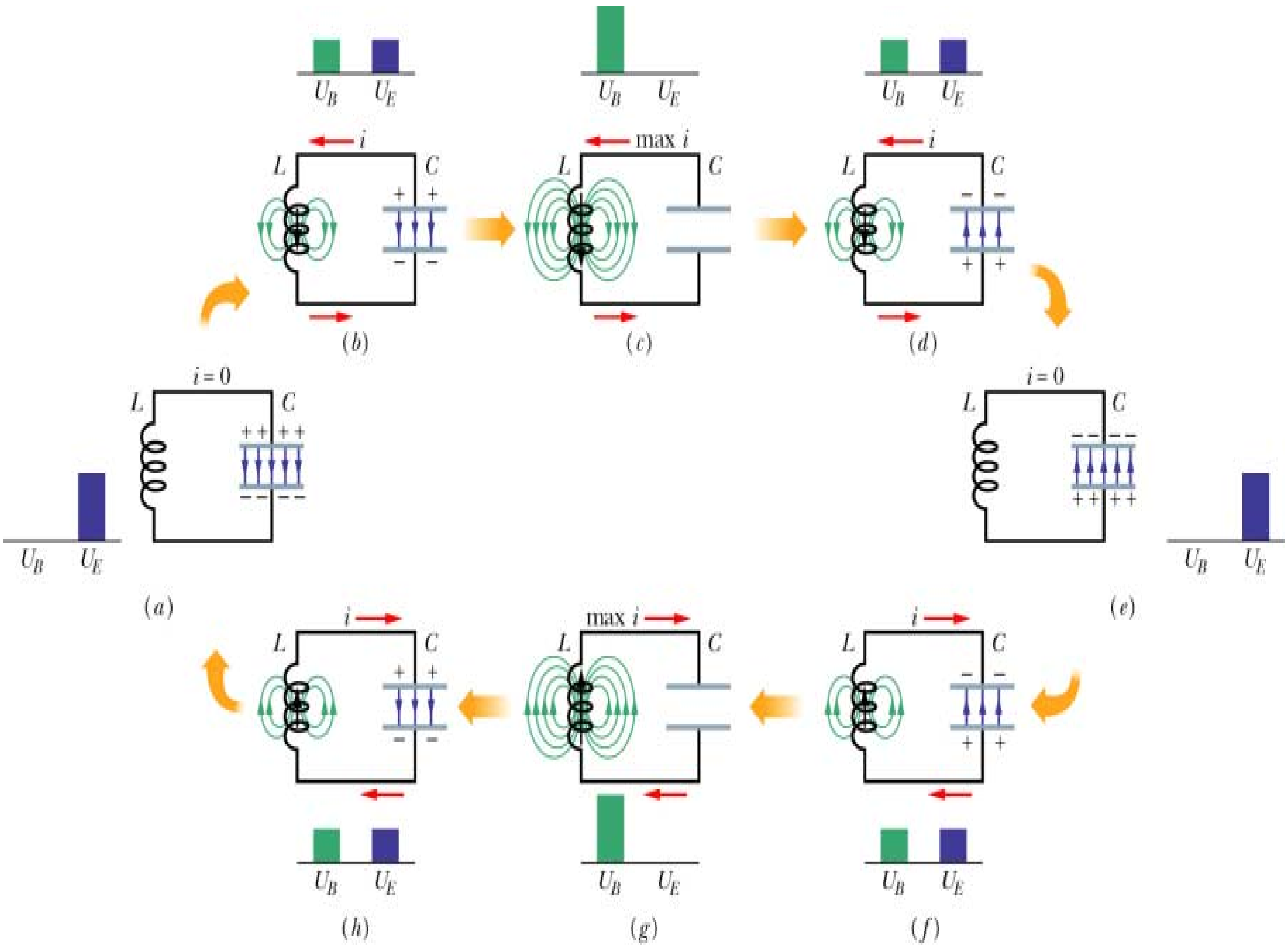


- Analogy to block-spring system (PHY183)

$$U = \tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2$$

$$\omega = \sqrt{\frac{k}{m}}$$





U_B U_E
i
max i
i = 0
L
C
(a)
(b)
(c)
(d)
(e)
(f)
(g)
(h)

LC Circuits (checkpoint #1)

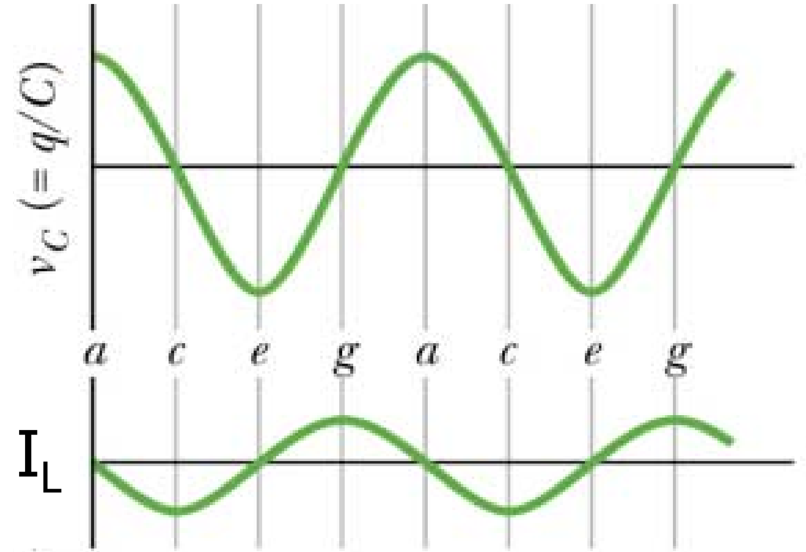
- A charged capacitor & inductor are connected in series at time t=0. In terms of period, T, how much later will the following reach their maximums:
  - *q* of capacitor

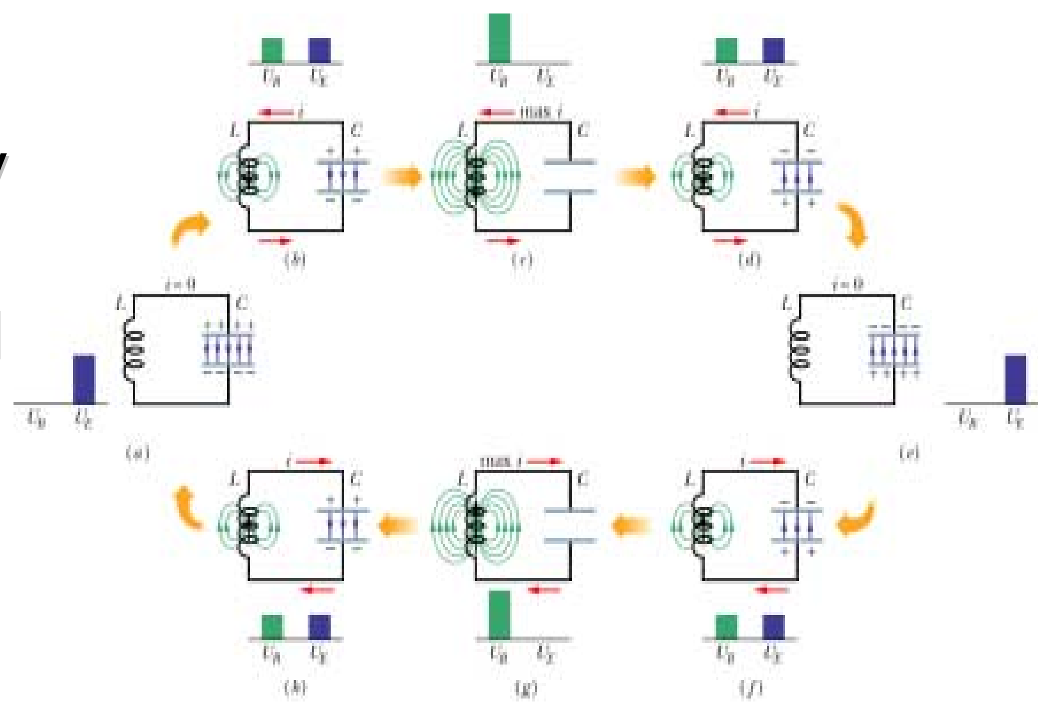
    T/2
  - V_c with original polarity

    T
  - Energy stored in *E* field

    T/2
  - The current

    T/4

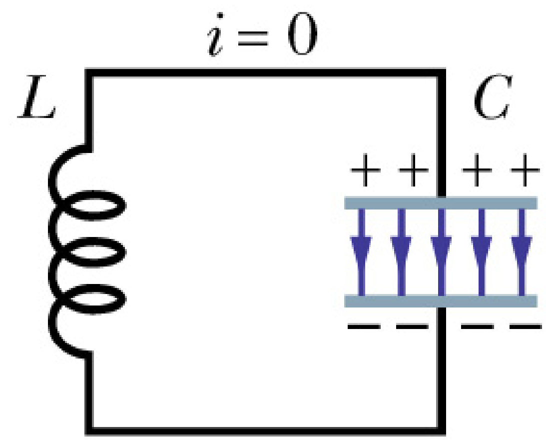




LC Circuits

- The phase constant, ϕ, is determined by the conditions at time t=0 (or some other time)

$$q = Q\cos(\omega t + \phi)$$

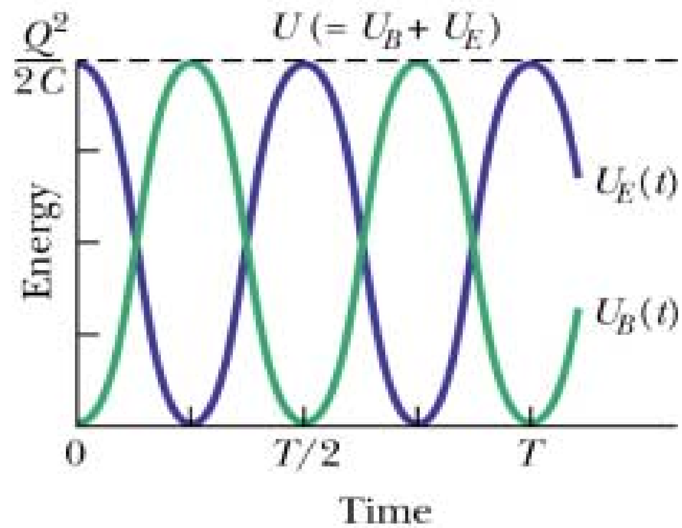


- If ϕ = 0 then at t = 0, $q = Q$
- Current is $i = -I\sin(\omega t)$
- Where $I = \omega Q$ and $\omega = \sqrt{\dfrac{1}{LC}}$

LC Circuits

$$U_E = \frac{Q^2}{2C}\cos^2(\omega t)$$

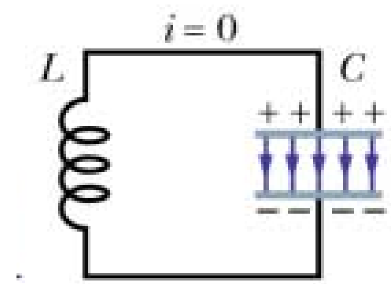
$$U_B = \frac{Q^2}{2C}\sin^2(\omega t)$$



- Maximum value for both $U_{E,\max} = U_{B,\max} = Q^2/2C$
- At any instant, sum is $U = U_B + U_E = Q^2/2C$
- When U_E = max, U_B = 0, and conversely, when U_B = max, U_E = 0

LC Circuits (checkpoint #2)-quiz

- Capacitor in LC circuit has $V_{C,max} = 15\ V$ and $U_{E,max} = 150\ J$. When capacitor has $V_C = 5\ V$ and $U_E = 50\ J$, what are the
  - 1) emf across the inductor?
  - 2) the energy stored in the B field?



- Apply the loop rule
  - Net potential difference around the circuit must be zero

$$v_L(t) = v_C(t)$$

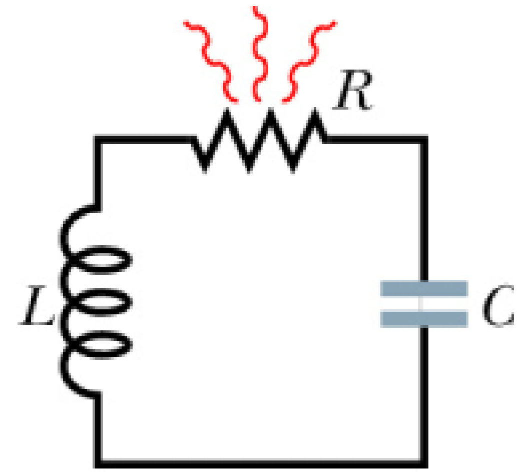
1) 5 V - answer B

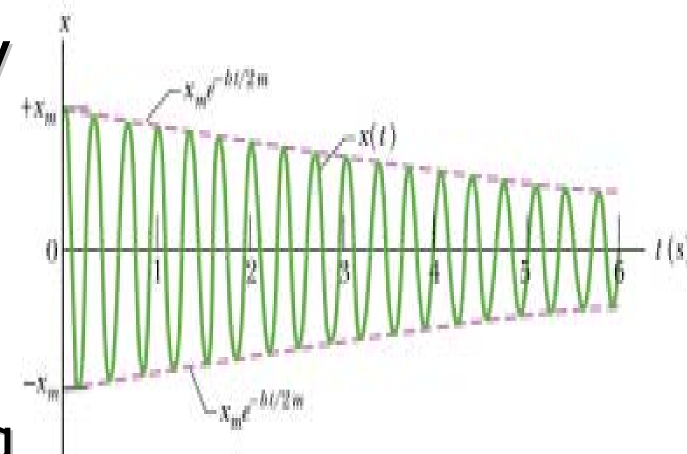
2) 100 J - answer C

$$U_{E,\max} = U_E(t) + U_B(t)$$

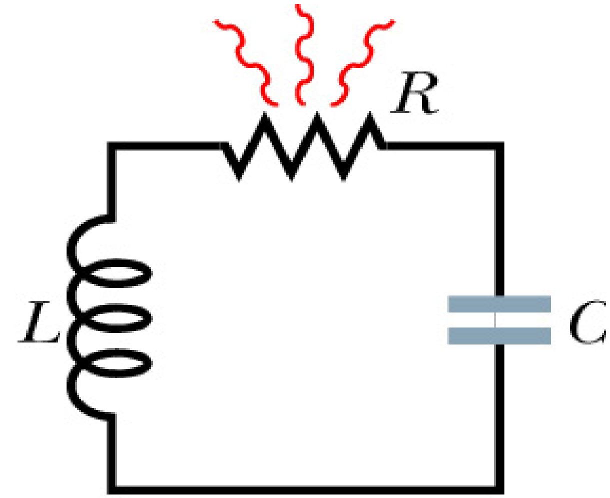
Damped oscillator

- RLC circuit – resistor, inductor and capacitor in series
- Total electromagnetic energy, $U = U_E + U_B$, is no longer constant
- Energy decreases with time as it is transferred to thermal energy in the resistor
- Oscillations in q, i and V are damped
  - Same as damped block and spring





RLC Circuits



- Resistor does not store electromagnetic energy so total energy at any time is

$$U = U_B + U_E = \frac{Li^2}{2} + \frac{q^2}{2C}$$

- Rate of transfer to thermal energy is (minus sign means U is decreasing)

$$\frac{dU}{dt} = -i^2 R$$

- Differentiating gives

$$\frac{dU}{dt} = Li\frac{di}{dt} + \frac{q}{C}\frac{dq}{dt} = -i^2 R$$

RLC Circuits

$$\frac{dU}{dt} = Li\frac{di}{dt} + \frac{q}{C}\frac{dq}{dt} = -i^2 R$$

- Use relations $i = \frac{dq}{dt}$ $\frac{di}{dt} = \frac{d^2q}{dt^2}$
- Differential equation for damped RLC circuit is

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0$$

- Solution

$$q = Qe^{-Rt/2L}\cos(\omega' t)$$

RLC Circuits

$$q = Qe^{-Rt/2L}\cos(\omega' t)$$

- Where $\omega' = \sqrt{\omega^2 - (R/2L)^2}$ $\quad \omega = \sqrt{\frac{1}{LC}}$
- Charge in RLC circuit is sinusoidal but with an exponentially decaying amplitude $Qe^{-Rt/2L}$
- Damped angular frequency, ω´, is always less than ω of the undamped oscillations
- If R is small enough can replace ω´ with ω

RLC Circuits

$$q = Qe^{-Rt/2L}\cos(\omega' t)$$

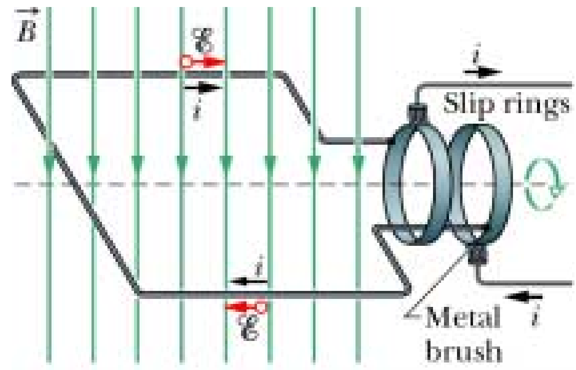
- Find U_E as function of time

$$U_E = \frac{q^2}{2C} = \frac{Q^2}{2C}e^{-Rt/L}\cos^2(\omega' t)$$

- Total energy decreases as $U_{tot} = \frac{Q^2}{2C}e^{-Rt/L}$

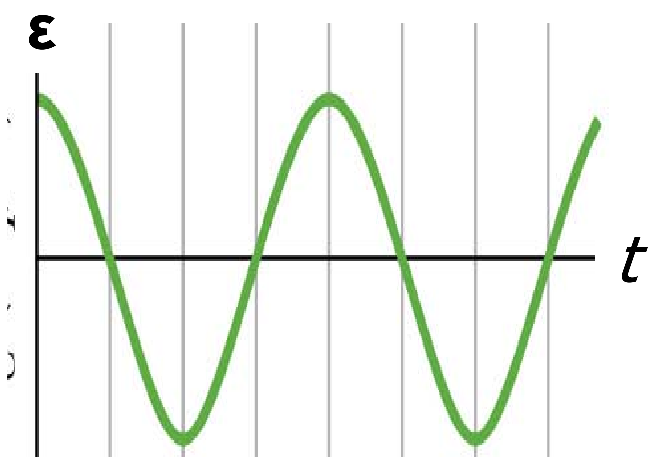
AC generator

- Mechanically turn loop in *B* field, induces a current and therefore an emf
- Driving angular frequency ω_d is equal to angular speed that loop rotates in *B* field.
- Used Faraday's law to find emf



$$\mathcal{E} = -N\frac{d\Phi_B}{dt} = NBA\omega_d \sin\omega_d t$$

$$\mathcal{E} = \mathcal{E}_m \sin\omega_d t \qquad \mathcal{E}_m = NBA\,\omega_d$$

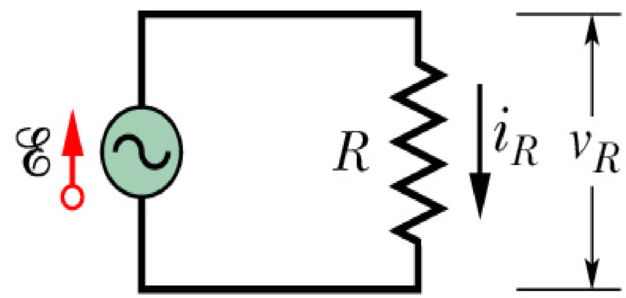


Forced Oscillations

Driving frequency ω_d will overpower the natural frequency ω

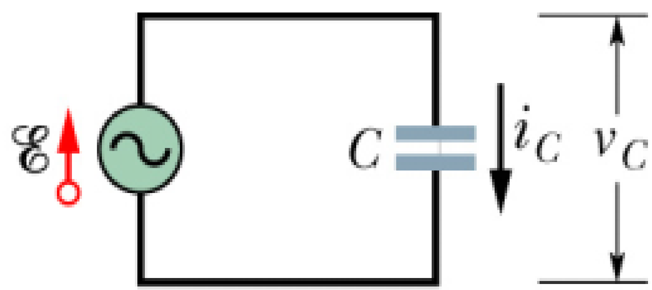
Resistive load

$$v_R = V_R \sin \omega_d t \qquad V_R = \mathcal{E}_m$$



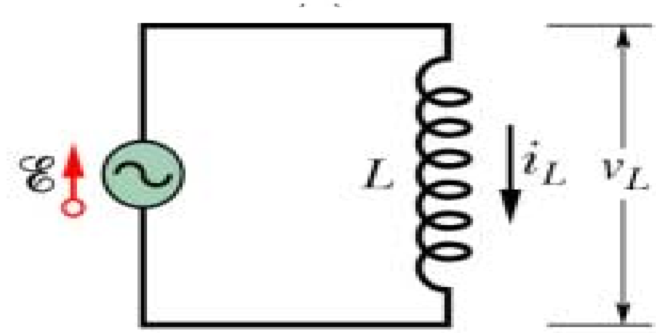
Capacitive load

$$v_C = V_C \sin \omega_d t \qquad V_C = \mathcal{E}_m$$



Inductive load

$$v_L = V_L \sin \omega_d t \qquad V_L = \mathcal{E}_m$$

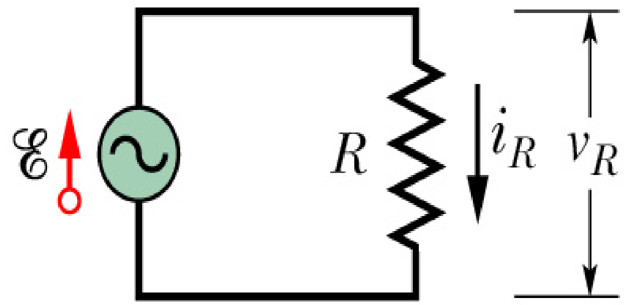


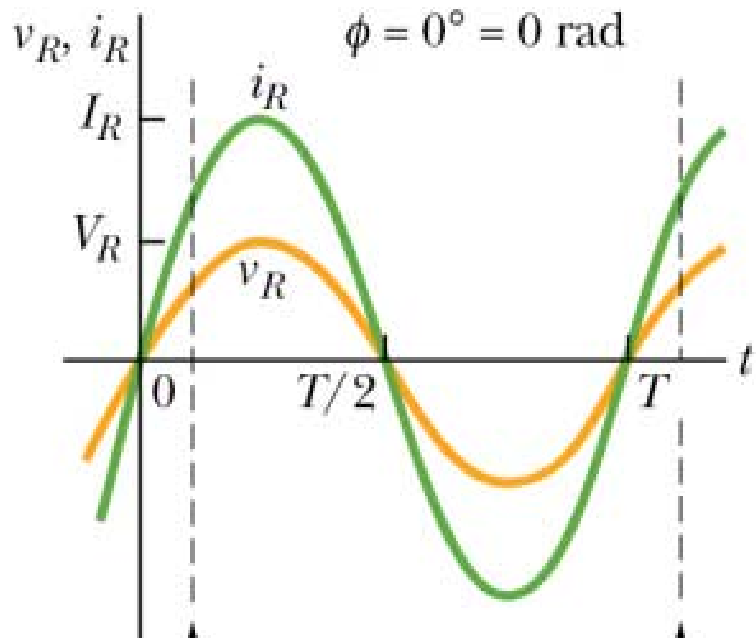
Forced oscillations - Resistive load

- Use definition of resistance to find i_R

$$v_R = V_R \sin \omega_d t$$

$$i_R = \frac{v_R}{R} = \frac{V_R}{R} \sin \omega_d t$$





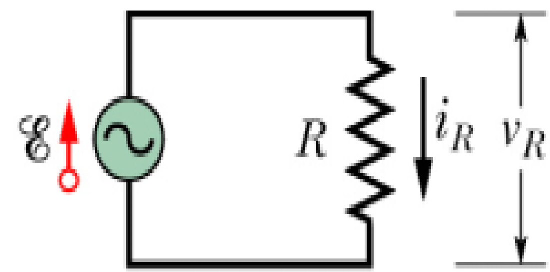
- Voltage and current are functions of $\sin(\omega_d t)$ with $\phi = 0$ so are in phase
- No damping of v_R and i_R, since the generator supplies energy

Forced oscillations - Resistive load

- Compare current to general form

$$i_R = \frac{v_R}{R} = \frac{V_R}{R}\sin\omega_d t$$

$$i_R = I_R \sin(\omega_d t - \phi)$$



- Minus sign for phase is tradition
- For purely resistive load the phase constant $\phi = 0$
- Voltage amplitude is related to current amplitude

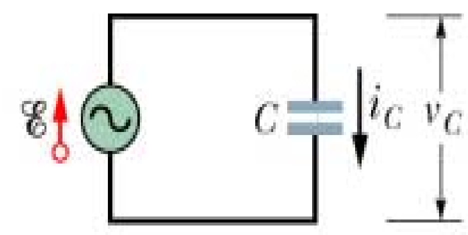
$$I_R = \frac{V_R}{R}$$

$$V_R = I_R R$$

Forced oscillations - Capacitive load

- Use definition of capacitance

$$q_C = Cv_C = CV_C \sin \omega_d t$$



- Use definition of current and differentiate

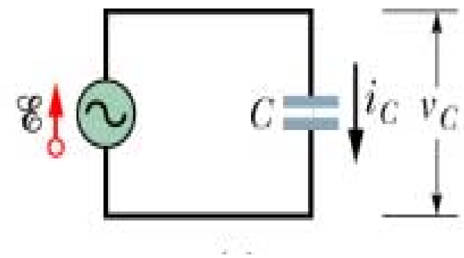
$$i_C = \frac{dq_C}{dt} = \omega_d C V_C \cos \omega_d t$$

- Replace cosine term with a phase-shifted sine term

$$\cos\omega_d t = \sin(\omega_d t + 90°)$$

Capacitive load

- Voltage and current relations



$$v_C = V_C \sin \omega_d t$$

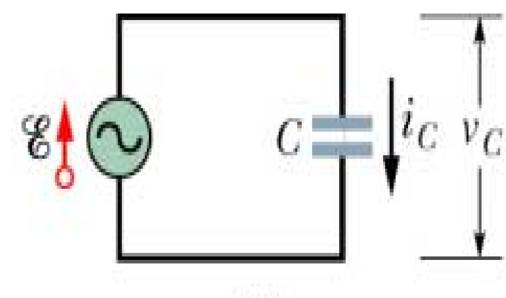
$$i_C = \omega_d C V_C \sin(\omega_d t + 90°) = I_C \sin(\omega_d t + 90°)$$

$$I_C = \frac{V_C}{X_C} \qquad X_C = \frac{1}{\omega_d C}$$

- X_C is called the capacitive reactance has units of ohms

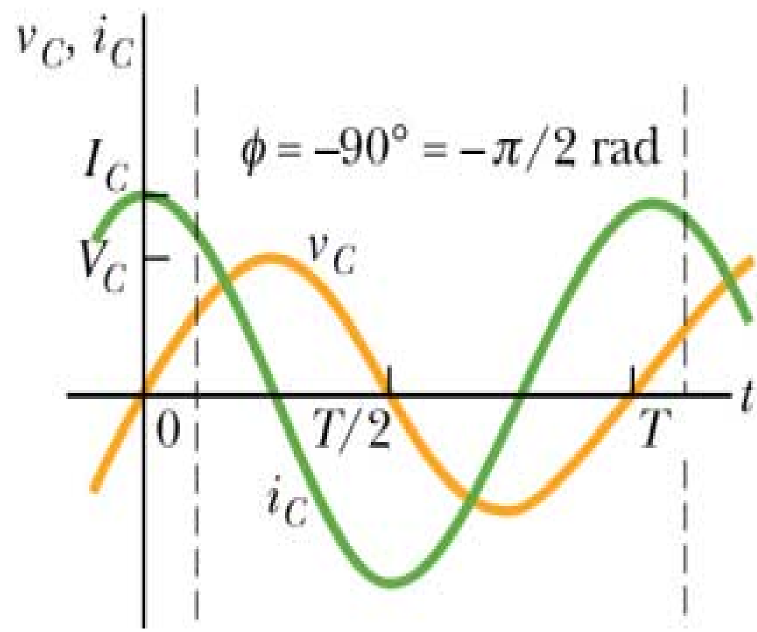
Forced oscillations - Capacitive load

- Compare v_C and i_C of capacitor



$$v_C = V_C \sin \omega_d t$$

$$i_C = \omega_d C V_C \sin(\omega_d t + 90^\circ)$$

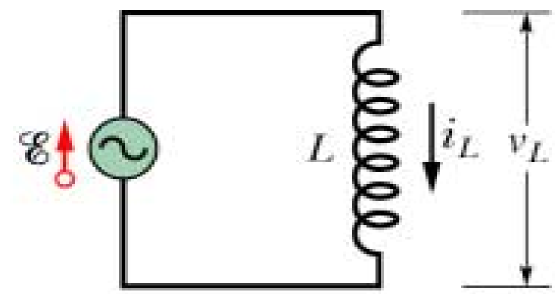


- Voltage and current are out of phase by 90°

- Current leads voltage by T/4

Forced Oscillations - Inductive load

- Voltage and current relations



$$v_L = V_L \sin \omega_d t$$

$$i_L = \frac{V_L}{\omega_d L}\sin(\omega_d t - 90°) = I_L \sin(\omega_d t - 90°)$$

$$I_L = \frac{V_L}{X_L}$$

$$X_L = \omega_d L$$

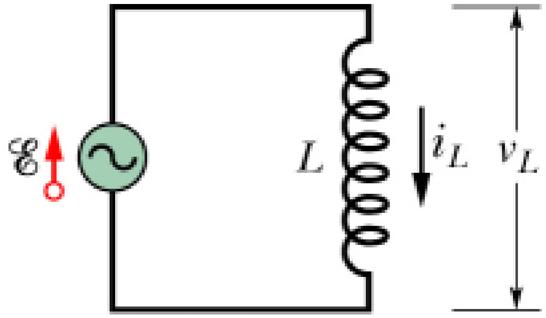
- X_C is called the inductive reactance has units of ohms

Forced Oscillations - Inductive load

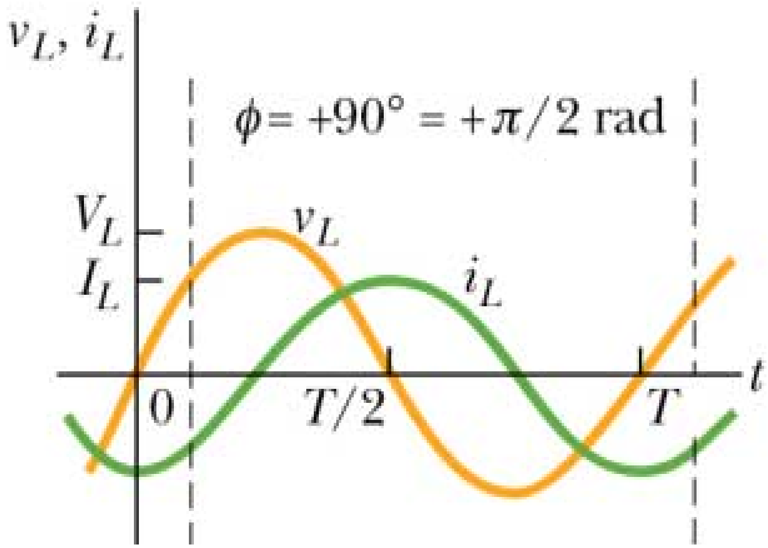
- Compare v_L and i_L of inductor

$$v_L = V_L \sin \omega_d t$$

$$i_L = I_L \sin(\omega_d t - 90°)$$



- Compare i_L to v_L
- i_L and v_L are 90° out of phase
- Current lags voltage by T/4



Summary of Forced Oscillations

Element	Reactance/ Resistance	Phase of Current	Phase angle ϕ	Amplitude Relation
Resistor	R	In phase	0°	$V_R = I_R R$
Capacitor	$X_C = 1/\omega_d C$	Leads v_C (ICE)	-90°	$V_C = I_C X_C$
Inductor	$X_L = \omega_d L$	Lags v_L (ELI)	+90°	$V_L = I_L X_L$

- ELI (positively) is the ICE man
  - Voltage or emf (E) before current (I) in an inductor (L)
  - Phase constant ϕ is positive for an inductor
  - Current (I) before voltage or emf (E) in capacitor (C)