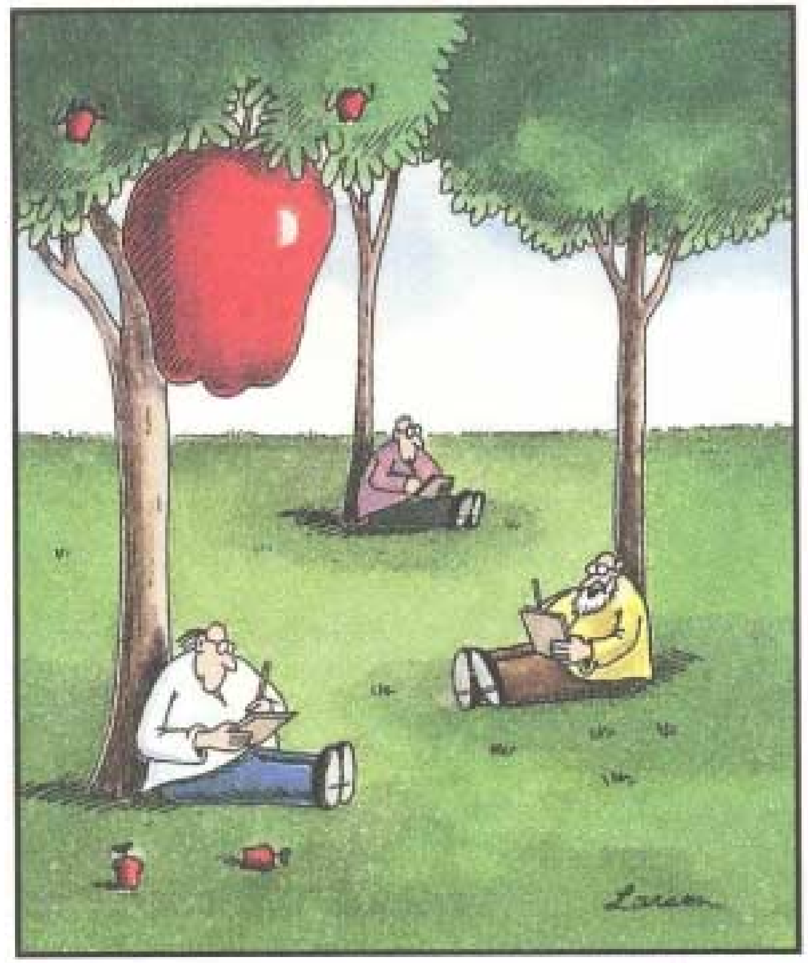
November 4th

Chapter 33

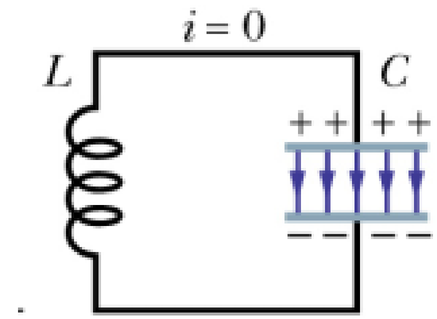
RLC Circuits



"Nothing yet. ... How about you, Newton?"

Review

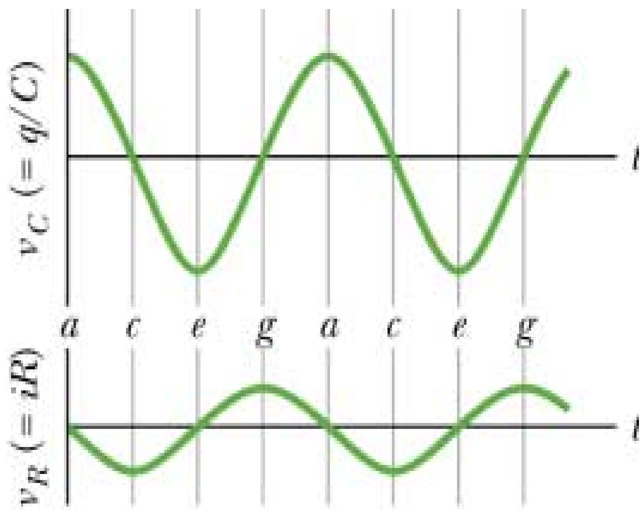
- RL and RC circuits
  - Charge, current, and potential grow and decay exponentially



- LC circuit
  - Charge, current, and potential change sinusoidally

  - Total electromagnetic energy is

$$U = U_B + U_E = \frac{Li^2}{2} + \frac{q^2}{2C}$$



Review

- Ideal LC circuit
  - Total energy conserved $$\frac{dU}{dt} = 0$$
  - Solved differential equation to find

$$q = Q\cos(\omega t)$$

$$i = -I\sin(\omega t)$$

$$\omega = \sqrt{\frac{1}{LC}}$$

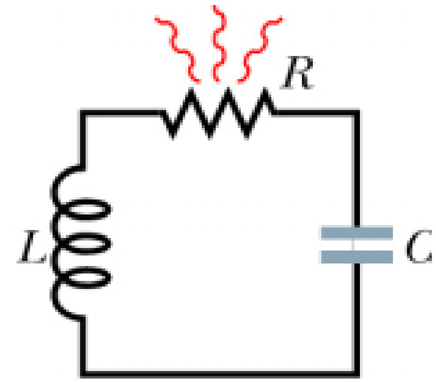
  - Substituting *q* and *i* into energy equations

$$U_E = \frac{Q^2}{2C}\cos^2(\omega t)$$

$$U_B = \frac{Q^2}{2C}\sin^2(\omega t)$$

$$U = U_B + U_E = Q^2/2C$$

Review



- RLC circuit
  - Energy is no longer conserved, becomes thermal energy in resistor
  - Oscillations are damped
  - Solved differential equation to find

$$\frac{dU}{dt} = -i^2 R$$

$$q = Qe^{-Rt/2L}\cos(\omega' t)$$

$$\omega' = \sqrt{\omega^2 - (R/2L)^2}$$

If *R* is very small $\omega' = \omega$

  - Energy goes as

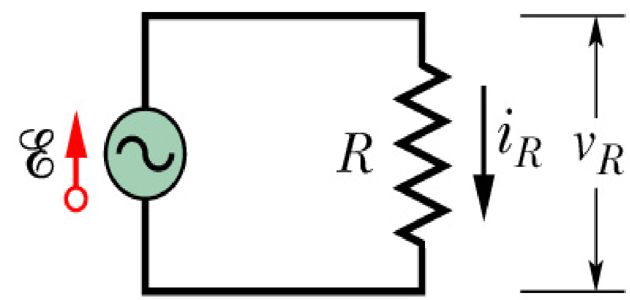
$$U_E = \frac{Q^2}{2C} e^{-Rt/L}\cos^2(\omega' t)$$

$$U_{tot} = \frac{Q^2}{2C} e^{-Rt/L}$$

Resistive Load

- Apply loop rule $\mathcal{E} - v_R = 0$

  $v_R = \mathcal{E}$



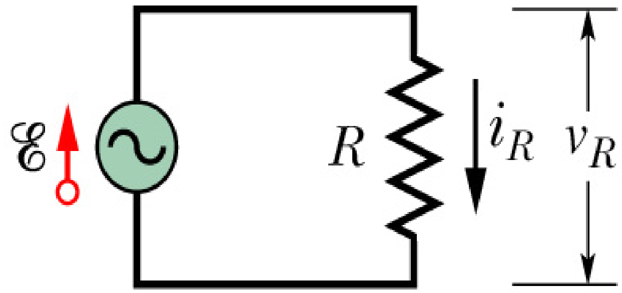
- Using $\mathcal{E} = \mathcal{E}_m \sin \omega_d t$
- We have $v_R = \mathcal{E}_m \sin \omega_d t$
- Amplitude across resistor is same as across emf $\mathcal{E}_m = V_R$
- Rewrite v_R as $v_R = V_R \sin \omega_d t$

Forced Oscillations

Resistive load

$$v_R = V_R \sin \omega_d t$$

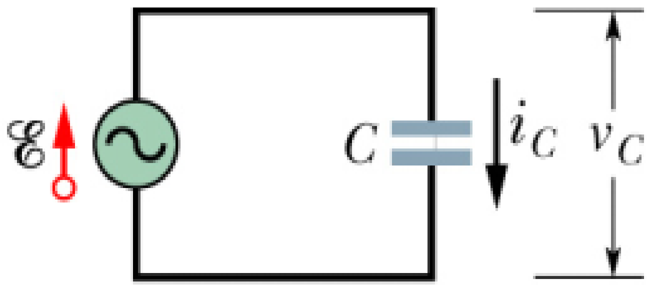
$$V_R = \mathcal{E}_m$$



Capacitive load

$$v_C = V_C \sin \omega_d t$$

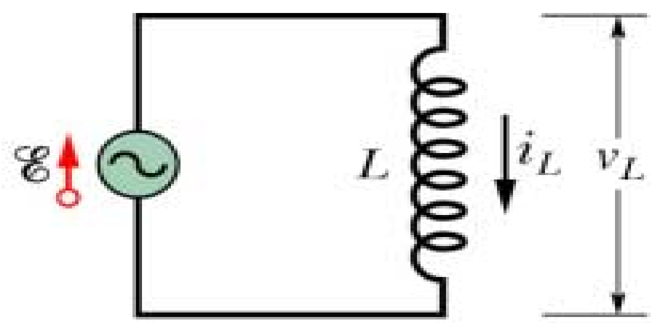
$$V_C = \mathcal{E}_m$$



Inductive load

$$v_L = V_L \sin \omega_d t$$

$$V_L = \mathcal{E}_m$$

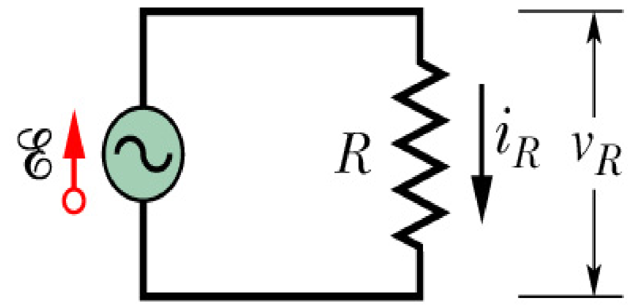


Resistive load

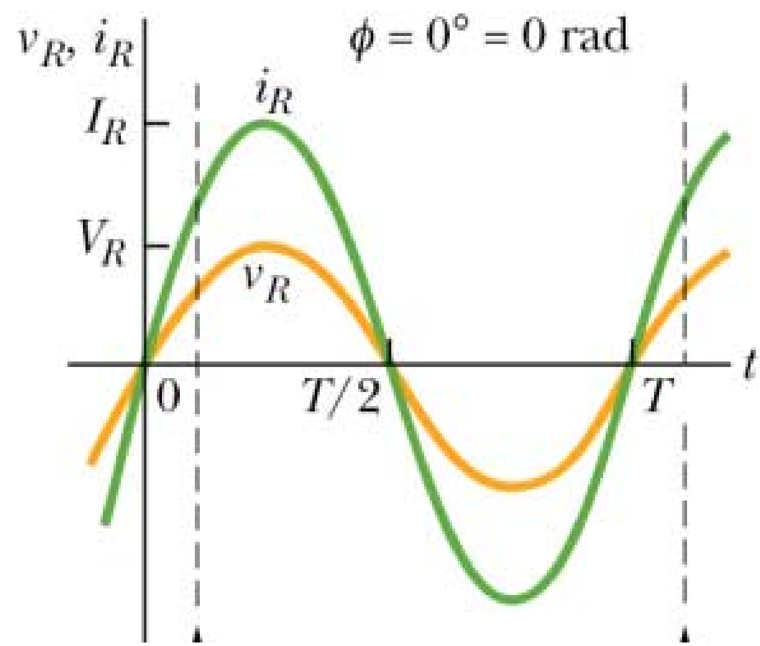
- Use definition of resistance to find i_R

$$v_R = V_R \sin \omega_d t$$

$$i_R = \frac{v_R}{R} = \frac{V_R}{R} \sin \omega_d t$$



- Voltage and current are functions of $\sin(\omega_d t)$ with $\phi = 0$ so are in phase
- No damping of v_R and i_R, since the generator supplies energy

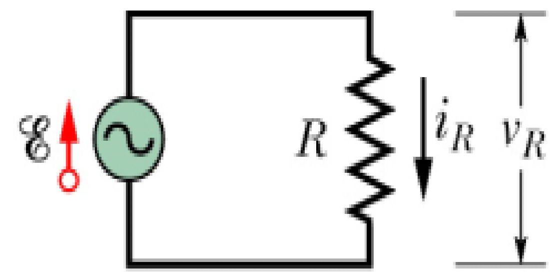


Resistive load

- Compare current to general form

$$i_R = \frac{v_R}{R} = \frac{V_R}{R}\sin\omega_d t$$

$$i_R = I_R \sin(\omega_d t - \phi)$$



- Minus sign for phase is tradition
- For purely resistive load the phase constant $\phi = 0$
- Voltage amplitude is related to current amplitude

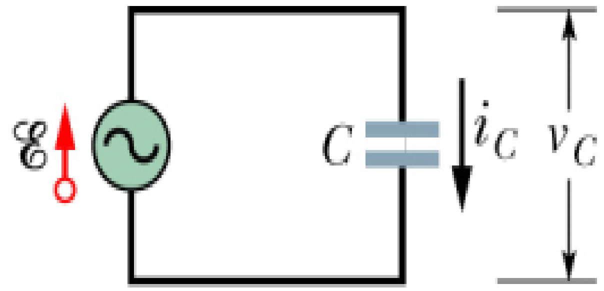
$$I_R = \frac{V_R}{R}$$

$$V_R = I_R R$$

Capacitive load

- Use definition of capacitance

$$q_C = Cv_C = CV_C \sin \omega_d t$$



- Use definition of current and differentiate

$$i_C = \frac{dq_C}{dt} = \omega_d CV_C \cos \omega_d t$$

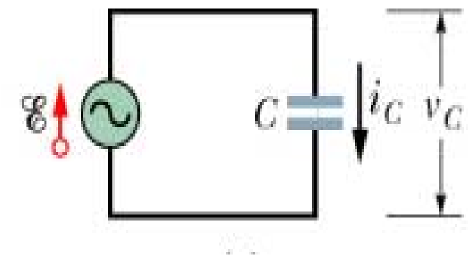
- Replace cosine term with a phase-shifted sine term

$$\cos\omega_d t = \sin(\omega_d t + 90°)$$

Capacitive load

- Voltage and current relations

$$v_C = V_C \sin \omega_d t$$



$$i_C = \omega_d C V_C \sin(\omega_d t + 90°) = I_C \sin(\omega_d t + 90°)$$

$$I_C = \frac{V_C}{X_C} \qquad X_C = \frac{1}{\omega_d C}$$

- X_C is called the capacitive reactance and has units of ohms

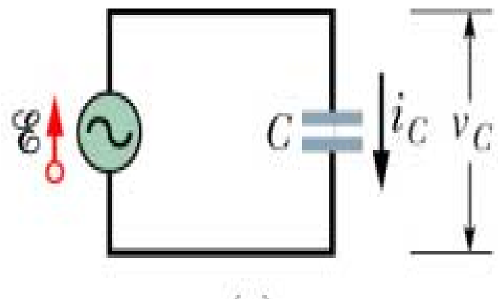
Capacitive load

- Compare v_C and i_C of capacitor

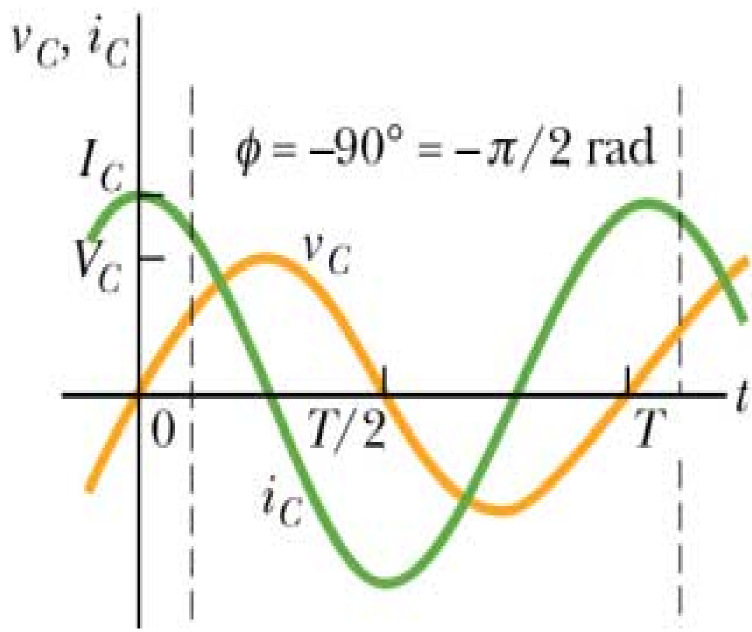
$$v_C = V_C \sin \omega_d t$$

$$i_C = \omega_d C V_C \sin(\omega_d t + 90°)$$

$$i_R = I_R \sin(\omega_d t - \phi)$$



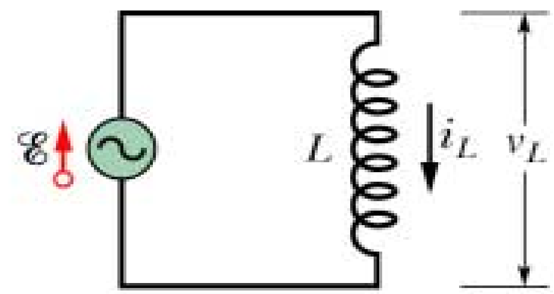
- Voltage and current are out of phase by -90°
- Current leads voltage by T/4



Inductive load (skipped in lecture)

- Derivation of current:
- Self-induced emf across an inductor is

$$\mathcal{E}_L = v_L = L\frac{di}{dt}$$



- Relate

$$v_L = V_L \sin\omega_d t = L\frac{di}{dt} \qquad \frac{di}{dt} = \frac{V_L}{L}\sin\ \omega_d t$$

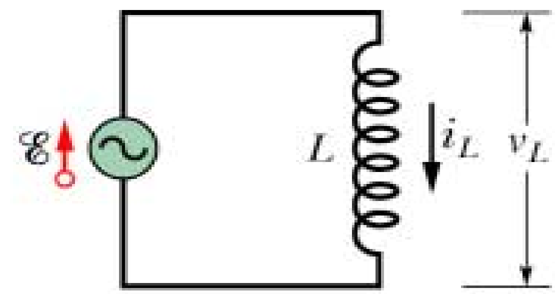
- Want current so integrate

$$i_L = \frac{V_L}{L}\int \sin(\omega_d t)\ dt = -\left(\frac{V_L}{\omega_d L}\right)\cos(\omega_d t)$$

- Then use $-\cos(\omega_d t) = \sin(\omega_d t - 90°)$

Inductive load

- Voltage and current relations



$$v_L = V_L \sin \omega_d t$$

$$i_L = \frac{V_L}{\omega_d L}\sin(\omega_d t - 90°) = I_L \sin(\omega_d t - 90°)$$

$$I_L = \frac{V_L}{X_L} \qquad X_L = \omega_d L$$

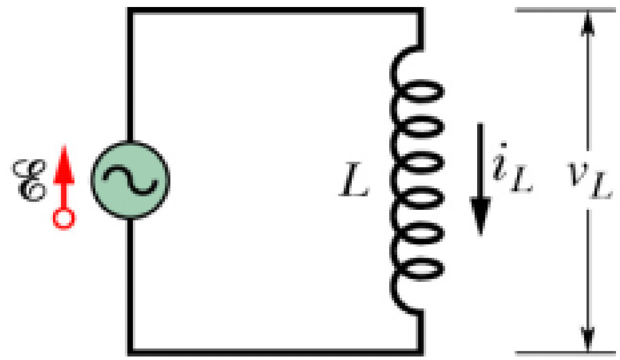
- X_L is called the inductive reactance has units of ohms

Inductive load

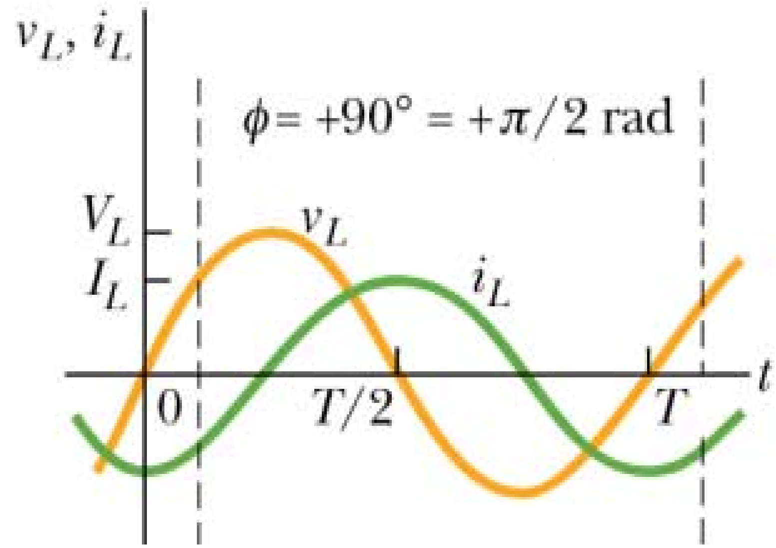
- Compare v_L and i_L of inductor

$$v_L = V_L \sin \omega_d t$$

$$i_L = I_L \sin(\omega_d t - 90°)$$



- i_L and v_L are +90° out of phase
- Current lags voltage by T/4



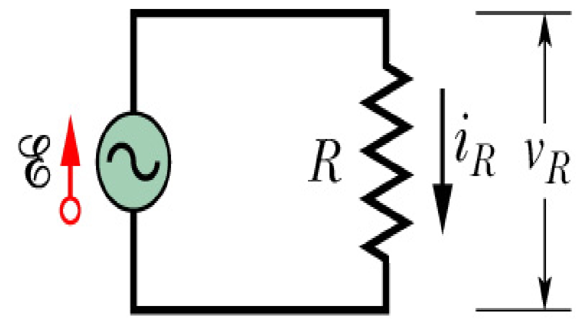
Summary of Forced Oscillations

Element	Reactance/ Resistance	Phase of Current	Phase angle ϕ	Amplitude Relation
Resistor	R	In phase	0°	$V_R=I_RR$
Capacitor	$X_C= 1/\omega_dC$	Leads v_C (ICE)	-90°	$V_C=I_CX_C$
Inductor	$X_L=\omega_dL$	Lags v_L (ELI)	+90°	$V_L=I_LX_L$

- ELI (positively) is the ICE man
  - Voltage or emf (E) before current (I) in an inductor (L)
  - Phase constant ϕ is positive for an inductor
  - Current (I) before voltage or emf (E) in capacitor (C)

EM Oscillations

- If the driving frequency, ω_d, in a circuit is increased does the amplitude voltage and amplitude current increase, decrease or remain the same?

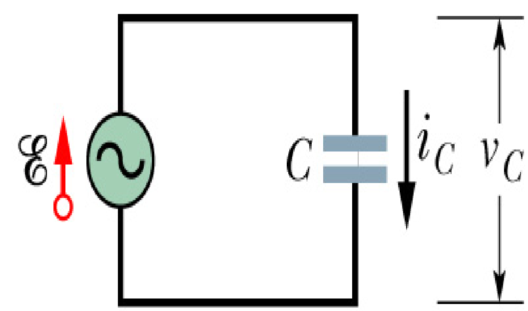


- For purely resistive circuit
- From loop rule $V_R = \varepsilon_m$
- So amplitude voltage, V_L stays the same
- I_R also stays the same -
  I_R only depends on R

$$I_R = \frac{V_R}{R}$$

EM Oscillations

- If the driving frequency, ω_d, in a circuit is increased does the amplitude voltage and amplitude current increase, decrease or remain the same?

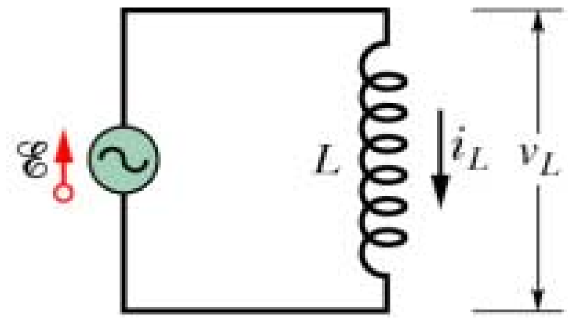


- For purely capacitive circuit
- From loop rule $V_c = \varepsilon_m$
- So amplitude voltage, V_C stays the same
- I_C depends on X_C which depends on ω_d by
- So I_C increases

$$I_C = \frac{V_C}{X_C} = \omega_d C V_C$$

EM Oscillations

- If the driving frequency, ω_d, in a circuit is increased does the amplitude voltage and amplitude current increase, decrease or remain the same?



- For purely inductive circuit
- From loop rule $V_L = \varepsilon_m$
- So amplitude voltage, V_L stays the same
- I_L depends on X_L which depends on ω_d by
- So I_L decreases

$$I_L = \frac{V_L}{X_L} = \frac{V_L}{\omega_d L}$$

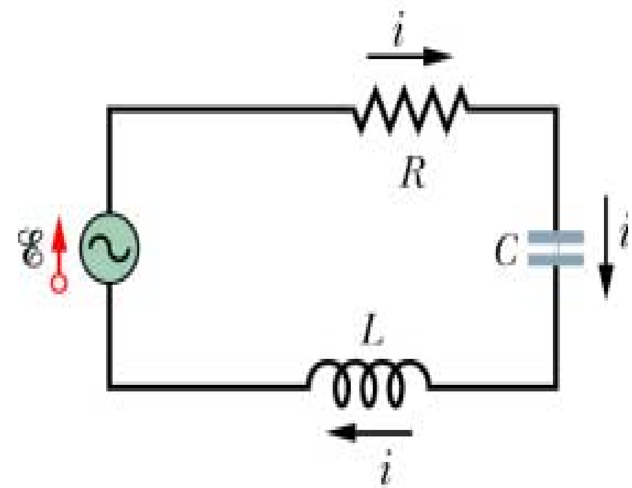
RLC Circuits

- LC and RLC circuits with no external emf
  - Free oscillations with natural angular frequency, ω

$$\omega' = \sqrt{\omega^2 - (R/2L)^2}$$

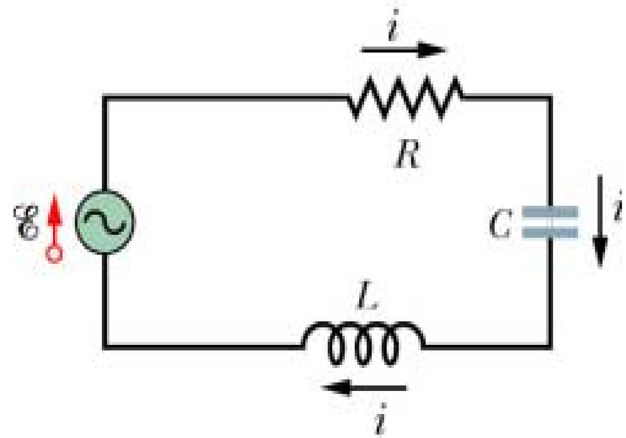
$$\omega = \sqrt{\frac{1}{LC}}$$

- Add external oscillating emf (e.g. ac generator) to RLC circuit
  - Oscillations said to be driven or forced
  - Oscillations occur at driving angular frequency, ω_d
  - When $\omega_d = \omega$, called resonance, the current amplitude, I, is maximum



RLC circuits

- RLC circuit – resistor, capacitor and inductor in series
- Apply alternating emf



$$\mathcal{E} = \mathcal{E}_m \sin \omega_d t$$

- Elements are in series so same current is driven through each

$$i = I \sin(\omega_d t - \phi)$$

- From the loop rule, at any time *t*, the sum of the voltages across the elements must equal the applied emf

$$\mathcal{E} = v_R + v_C + v_L$$

RLC circuits

$$\mathcal{E} = iR + iX_C + iX_L$$

- Taking into account the phase find current amplitude to be

$$I = \frac{\mathcal{E}_m}{\sqrt{R^2 + (X_L - X_C)^2}}$$

- Write amplitude voltage as

$$\mathcal{E}_m = IZ$$

- Where Z is the impedance
  - Like resistance and has units of Ohms

$$X_L = \omega_d L$$

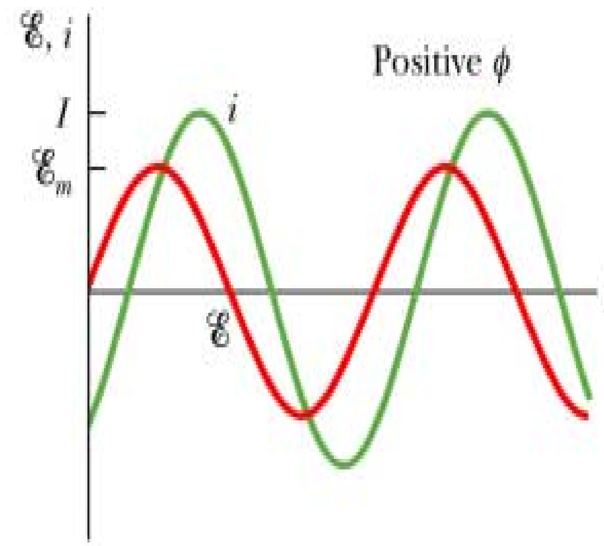
$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$

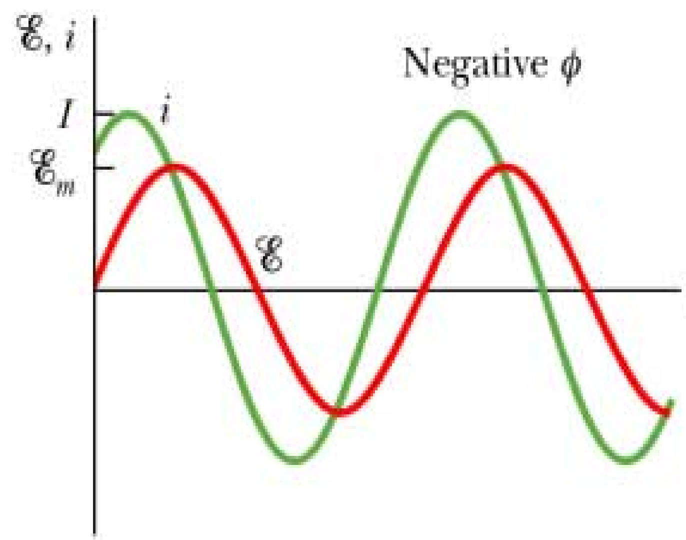
$$X_C = \frac{1}{\omega_d C}$$

EM Oscillations

$$\tan\phi = \frac{X_L - X_C}{R}$$

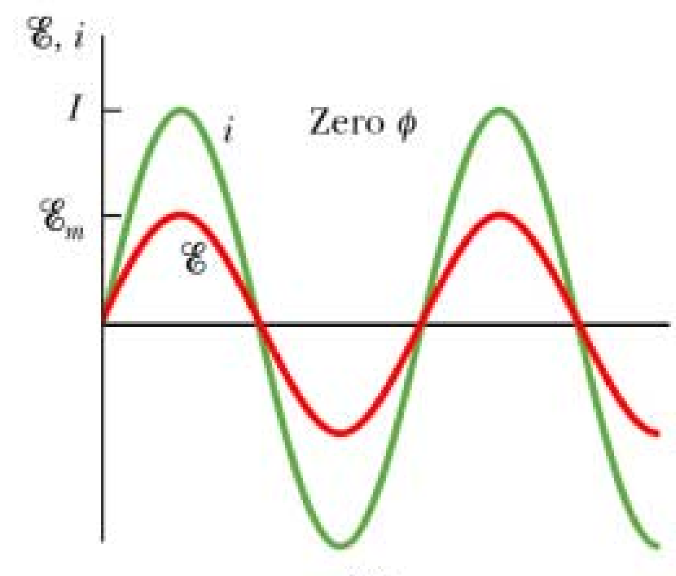
- If $X_L > X_C$ the circuit is more inductive than capacitive
  - ϕ is positive
  - Emf is before current (ELI)
- If $X_L < X_C$ the circuit is more capacitive than inductive
  - ϕ is negative
  - Current is before emf (ICE)





RLC Circuits

$$\tan\phi = \frac{X_L - X_C}{R}$$



- If $X_L = X_C$ the circuit is in resonance – emf and current are in phase
- Current amplitude I is max when impedance, Z is min

$$I = \frac{\mathcal{E}_m}{Z} = \frac{\mathcal{E}_m}{\sqrt{R^2 + (X_L - X_C)^2}} = \frac{\mathcal{E}_m}{R}$$

$$X_L - X_C = 0$$

EM Oscillations

- When $X_L = X_C$ the driving frequency is

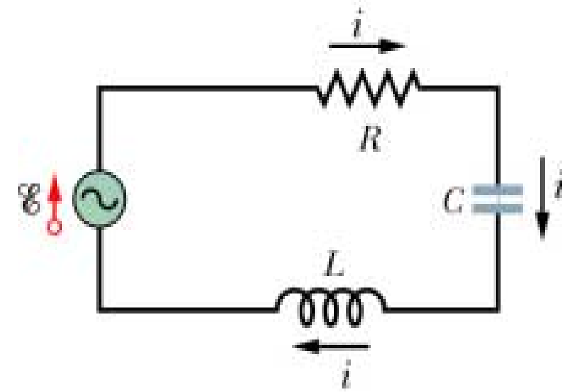
$$\omega_d L = \frac{1}{\omega_d C} \qquad \omega_d = \frac{1}{\sqrt{LC}}$$

- This is the same as the natural frequency, ω

$$\omega_d = \omega = \frac{1}{\sqrt{LC}}$$

- For RLC circuit, resonance and the max current I occurs when $\omega_d = \omega$

RLC circuits



$$I = \frac{E_m}{\sqrt{R^2 + \left(\omega_d L - \frac{1}{\omega_d C}\right)^2}}$$

- I is largest when

$$X_L = X_C$$

- When $\omega_d = \omega$ circuit said to be in resonance
  - When homework says resonance frequency it means $\omega_d = \omega$

$$\omega = \omega_d = 2\pi f_d$$