Experiment 3

The Simple Pendulum

Reading and problems (1 point for each problem)

Taylor, Sections 5.1, 5.2, 5.3 and 5.4; Problems 4.16, 4.23, 5.4 and 5.18.

Goals

1. To improve measurement accuracy by averaging
2. To study the amplitude and mass dependence of the period of a pendulum
3. To measure g with the simple pendulum
4. To study energy conservation
5. To examine the propagation of error in derived physical quantities

Theoretical Introduction

Part A

The simple pendulum shown below consists of an object of mass m suspended from a pivot by a massless string. The distance from the point of pivot to the center of mass of the ball, or "bob", is designated by L in the figure. When the ball is displaced from its resting position the string makes an angle Θ with the vertical. The component of the gravitational force in the tangential x-direction acts to restore it to its equilibrium position. Thus the restoring force is:

$$F_x = -mg\sin\Theta \qquad (1)$$

The tension of the string T, in the direction toward the point of suspension, is equal in magnitude and opposite in direction to the component of the gravitational force acting in that direction.

The mass is accelerated only in the tangential direction perpendicular to the string. Using Newton' s Second Law, F= ma, and $\Delta x = L\sin\Delta\Theta \cong L\Delta\Theta$ (see Eq. 5), the relation between the small tangential displacement Δx and the corresponding change in angle ΔΘ, one finds:

$$F_x = m.a_x = m\cdot\frac{d^2x}{dt^2} = m\cdot\frac{d^2}{dt^2}L\cdot\Theta = mL\frac{d^2\Theta}{dt^2} \qquad (2)$$

Imagine a pendulum with a mass at the end of a string of length L swinging by an angle Q from the vertical.

Inserting the expression for the restoring force, the equation of motion becomes:

$$mL\frac{d^2\Theta}{dt^2} = -mg\sin\Theta \qquad (3)$$

or

$$\frac{d^2\Theta}{dt^2} + \frac{g}{L}\sin\Theta = 0 \qquad (4)$$

For small angles Θ, we may expand sinΘ as follows

$$\sin\Theta \approx \Theta - \frac{\Theta^3}{6} + \frac{\Theta^5}{120} \qquad (5)$$

To simplify Eq. 4 we assume Θ is small and keep only the first term to obtain:

$$\frac{d^2\Theta}{dt^2} + \frac{g}{L}\Theta = 0 \qquad (6)$$

where the solution to differential equation (6) is:

$$\Theta = \Theta_o \sin(\sqrt{\frac{g}{L}} \cdot t) \qquad (7)$$

Because the sine function repeats itself whenever its argument changes by 2π, the time for one period T is given by:

$$\sqrt{\frac{g}{L}} \cdot T = 2p \qquad (8)$$

or

$$T = 2p\sqrt{\frac{L}{g}} \qquad (9)$$

which yields:

$$g = 4p^2 \frac{L}{T^2} \tag{10}$$

Thus as long as the small angle approximation of Eq. 5 is valid, the period is independent of the amplitude Θ_o and mass m. Measurement of the period T and the length L permit a determination of the gravitational constant g. If Θ_o is not small enough, Eq. 6 will not be valid, the period will depend on Θ_o, and will increase with the amplitude (see Appendix A).

Part B

For a pendulum swinging back and forth, the mechanical energy E shifts between kinetic and potential energy but remains constant (if no damping occurs):

$$E = K + U \tag{11}$$

$$U = mgh \tag{12}$$

$$K = \frac{1}{2}mv^2 \tag{13}$$

Here, h is the vertical displacement from its equilibrium position and v is the velocity of the bob. When the bob is at its maximum amplitude, all the energy is potential and v = 0. The bob has greatest speed at its lowest point, hence all the energy is kinetic and $U = 0$. Conservation of mechanical energy for these two instants is:

$$K_o + U_o = K_{max} + U_{max} \tag{14}$$

where the subscript o denotes the static equilibrium position and max stands for maximum point of oscillation.

Therefore, $\frac{1}{2}mv^2 + 0 = 0 + mgh$. (15)

Solving Eq. (15) for v^2 we obtain

$$v^2 = \left(\frac{\Delta x}{\Delta t}\right)^2 = 2gh. \tag{16}$$

Experimental Procedures

Part A

A bob is suspended from a pivot by a string. A protractor below the pivot allows us to set the initial pendulum amplitude by setting the angle. You will measure the period of the pendulum by:

1. Manual timing with a digital clock
2. Automatic timing with a photogate timer.

You then should be able to compute g from the period of the pendulum and the pendulum length. Measure the length L between the pivot of the pendulum and the center of mass of the bob as accurately as possible. You may need several measurements, or measurement strategies, to find L. Assess the role of statistical and systematic uncertainties in your value for L.

Questions for preliminary discussion

1. Verify by direct substitution that Eq. (7) is a solution of Eq. (6).
2. Should the period of the pendulum depend on the mass?
3. Should the period of the pendulum depend on its amplitude?
4. What is the criterion for a small amplitude oscillation?
5. Draw a diagram of forces acting on the bob.
6. What component of the force causes oscillation?
7. Discuss the sources of systematic and random errors in this experiment.

Manual measurement of T

It is most accurate to begin timing the swing of the pendulum at its lowest point because then the ball moves most quickly and takes the least time to pass by. The amplitude of the swing should be large in order to increase the speed of the pendulum at that point. On the other hand, Θ must be kept small enough that the approximation $\sin\Theta_o \cong \Theta_o$ remains valid. As a compromise, take the initial amplitude to be about 0.1 radian ($\sim 6^o$). Indicate on your data sheet your calculation to set Θ_o to 0.1 radian.

1. Using the timer, measure the duration of 25 *complete* cycles **ten times**, starting and stopping the measurements at the lowest point of the swing.
2. From these ten measurements, calculate the period (mean) and the standard deviation of the mean. You should enter the data into *Kgraph*. Do not round off your numbers too early in your calculations or you may lose accuracy in your final result. Calculate g and its uncertainty with the use of Eq. 10.

Automatic Measurement of T

1. Set the photogate on PEND position. Practice timing the period using the photogate a few times. How many periods does the photogate measure? From three measurements of the period with the photogate, calculate the mean period and the standard deviation of the mean. Calculate g and its uncertainty.
2. Change the bob and find the period for three other masses. Are your results consistent with Eq. 10?
3. Reduce the length of the string to 1/2, 1/3, 1/4, and 1/5 of its length by moving the middle support. Measure the period and perform a least-squares fit to the data by plotting T^2 vs. L. Find g and its uncertainty.
4. An oscillating solid rod with uniform cross-section also forms a pendulum. If suspended at one end, its period is given by $T = 2\pi\sqrt{\frac{2L}{3g}}$. Repeat the procedure above in (1) for the solid rod pendulum. See Appendix B for a derivation of this result.

Amplitude dependence of the period

If the amplitude of oscillation of a pendulum is not sufficiently small, its period will depend on amplitude. Thus Eq. 9 will not be valid. See Appendix A for a brief discussion.

Using the photogate timer, measure the period of the pendulum for a series of initial angles. Begin with 30° (about 0.5 radians) maximum and repeat for approximately $25^o, 20^o, 15^o, 10^o, 6^o$. Calculate the ratio $T(\Theta)/T_o$, where $T(\Theta)$ is defined by Eq. A3 and

T_o is the small amplitude period. Perform a linear least-squares fit of the data to T/T_o vs. Θ^2 (Θ in rad). Compare your results with the theoretical value A = 1/16 from $\frac{T}{T_o} = 1 + A\Theta^2$. It is not necessary to do an extended uncertainty analysis here.

Part B

Conservation of Energy

In this experiment, we will test the idea of conservation of energy by measuring the velocity of the bob (kinetic energy) as a function of its release height (potential energy). Measure the diameter of the bob with maximum accuracy. Set the photogate in *Gate* position. Determinethe vertical displacement from equilibrium h for the angles $\Theta = 30^o$ to 5^o at 5^o intervals. When the bob passes the equilibrium point, the photogate timer measures the time interval over which the bob interrupts the light. Calculate v_o and verify that $v_o^2 = 2gh$. Plot $1/\Delta t^{\ 2}$ vs. h (Eq. 16). Perform a linear least-squares fit to the data to obtain g. It is not necessary to do an extended uncertainty analysis here, either.

Phase Space Portrait

When the bob is released from an angle Θ, the pendulum oscillates between $+\Theta$ and $-\Theta$. In one cycle, the bob passes each point twice. The momentum p at these two instants is $p = +mv$ and $p = -mv$. Make a *phase space* plot with the orthogonal axes p and Θ. For example, when $\Theta = 30^o, U = mgh$ and $p = 0$. When $\Theta = 0^o$, $p = -mv$ (left direction as negative); at $\Theta = -30^o$, $p = 0$; and at $\Theta = 0^o$, p = +mv. Connect these four points with smooth line, forming an ellipse. Repeat for smaller angles. Now, imagine that the pendulum's amplitude is continuously decreasing to zero. Sketch the diagram for this situation. For a pendulum that is being damped, the diagram encompasses its complete dynamics, from start to finish. Explain this in your report.

Damped Pendulum

Finally, we consider the damped pendulum. Here, frictional losses decrease the energy of the pendulum as a function of time. To find the functional form of the energy decay, you will measure the peak velocity of the pendulum vs. time. Set L to about 20 cm and the photogate timer to GATE with MEM off. Take a measurement of the bob velocity at 50 s intervals, determined by reading the manual timer. Take at least 10 readings.

Calculate E and plot E vs. t .Can you make a transformation so that a straight line results? Can you define a characteristic time for the energy decay? Explain the graph in your report.

Questions

Address the following questions in your report:

1. What is the point of measuring 250 cycles (25 x 10) in Part A? Would it have been as accurate to measure one cycle 250 separate times?
2. Which value of g is more accurate, the one obtained by hand-timing or obtained with the photogate?
3. Assuming that your uncertainties are random, how many hand-timing measurements should be done to make the two sets of measurements equally precise?
4. Which quantity, T or L, makes the larger contribution to the fractional uncertainty in g? Does this suggest a way to improve the experiment?
5. Compare g with 9.804 m/s^2. Do you have a significant discrepancy? Discuss this quantitatively from a statistical point of view.
6. In part B, what might influence your experiment, and verification of $v^2 = 2gh$? Compare your calculation for g with the accepted value.
7. What is the major source of error in part B in testing energy conservation?

Appendix A

The Finite Amplitude Pendulum

To solve Eq.4 exactly,

$$\frac{d^2\Theta}{dt^2} + \frac{g}{L}\sin\Theta = 0 \ , \qquad \text{(A1)}$$

one may formally write the solution for the period T as an integral over the angle Θ:

$$T = 2\sqrt{\frac{L}{g}}\int_0^{\Theta_o}\left[\left(\sin\frac{\Theta}{2}\right)^2 - \left(\sin\frac{\Theta_o}{2}\right)^2\right]^{-\frac{1}{2}} d\Theta \qquad \text{(A2)}$$

Integrals of this form belong to a class of elliptic integrals which do not have closed form solutions. If the angle Θ_o is sufficiently small, solutions to any desired degree of accuracy can be obtained by doing series expansions in the angles. The result is:

$$T = 2p\sqrt{\frac{L}{g}}\left(1 + \frac{1}{16}\Theta_o^2 + \frac{11}{3072}\Theta_o^4 + \cdots\right) \qquad \text{(A3)}$$

Note that the period *increases* as the amplitude *increases*.

Appendix B

Solid Rod Pendulum

For a physical pendulum like a rod pivoting at one end, a restoring torque t reduces the angle when the pendulum is displaced from its equilibrium.

$$t = -(mg\sin\Theta)(h) \qquad \text{(B1)}$$

where $mg\sin\Theta$ is the tangential component of the force and h is the distance from the pivot to the center of mass. For small angles Eq. B1 becomes:

$$t \approx -(mgh)\Theta \qquad \text{(B2)}$$

which is the angular form of the Hooke's law. Writing the differential form of the torque and setting it equal to Eq. B2, we obtain the equation of motion:

$$t = rF = rm\frac{d^2(r\Theta)}{dt^2} = mr^2\frac{d^2\Theta}{dt^2} = -(mgh)\Theta \qquad \text{(B3)}$$

or

$$\frac{d^2\Theta}{dt^2} = -(\frac{mgh}{I})\Theta \qquad \text{(B4)}$$

where $I = \Sigma mr^2$ is the moment of inertia. The solution to (B4) is :

$$\Theta = \Theta_o \sin(\sqrt{\frac{mgh}{I}} \cdot t) \qquad \text{(B5)}$$

so that the time for one period is

$$T = 2p\sqrt{\frac{mgh}{I}}\,. \qquad \text{(B6)}$$

For the rod $I = \frac{1}{3}mL^2$ and the center of mass distance h = L/2 where L is the length and m is the mass of the rod. Therefore, we find that

$$T = 2p\sqrt{\frac{2l}{3g}}\,. \qquad \text{(B7)}$$