Small oscillations

In many encountered situations, systems oscillate around positions of equilibrium. If the oscillations are small enough, the motion can be described as that for a system of simple harmonic oscillators.

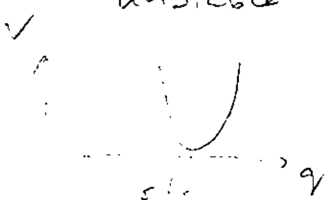
Lagrangian $_{L=T+V}$ w/o explicit time dependence

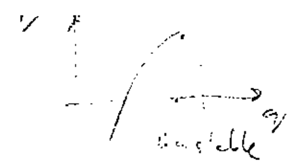
Position of equilibrium: One at which all generalized forces vanish

$$Q_i = -\frac{\partial V}{\partial q_i}\Big|_{q_0} = 0$$

– characterized by $q_{10} \ldots q_{n0}$

Equilibrium position stable, if a small disturbance produces motion bounded to the vicinity of the equilibrium point. It is unstable, if ~~a disturbance~~ a small displacement can produce an unbounded motion. The equilibrium will be unstable if V decreases in any direction from q_0.





Expanding potential around the equilibrium point:

$$V(q_1 \ldots q_n) = V(q_{10} \ldots q_{n0}) + \underbrace{\frac{\partial V}{\partial q_i}\Big|_{q_0}}_{=0} \eta_i + \frac{1}{2}\frac{\partial^2 V}{\partial q_i \partial q_j}\Big|_{q_0} \eta_i \eta_j + \ldots$$

$\eta_i = q_i - q_{i0}$

For stability, the matrix of 2nd derivatives must be positive definite, i.e. for any direction vector $\vec{n}$: ↓ in n dim

(necessary but not sufficient condition)

$$\frac{\partial^2 V}{\partial q_i \partial q_j} n_i n_j \geq 0$$

↑ $= 0$ along any direction may indicate a neutral equilibrium (V flat)

> 0 along any direction a sufficient condition for stability

$$V = V_0 + \frac{1}{2} V_{ij} \eta_i \eta_j$$

where $$V_{ij} = \left.\frac{\partial^2 V}{\partial q_i \partial q_j}\right|_0 = V_{ji}$$

Kinetic energy

$$T = \frac{m_\ell}{2} \dot{x}_\ell^2 = \frac{m_\ell}{2} \frac{\partial x_\ell}{\partial q_i} \frac{\partial x_\ell}{\partial q_j} \dot{q}_i \dot{q}_j$$

$$= \frac{1}{2} m_{ij} \dot{q}_i \dot{q}_j$$

$$m_{ij} = m_\ell \frac{\partial x_\ell}{\partial q_i} \frac{\partial x_\ell}{\partial q_j} = m_{ji}$$

The mass matrix is symmetric and positive definite

$$m_{ij} n_i n_j > 0$$

$$m_\ell \frac{\partial x_\ell}{\partial q_i} \frac{\partial x_\ell}{\partial q_j} n_i n_j = m_\ell \left(\frac{\partial x_\ell}{\partial q_i} n_i\right) \left(\frac{\partial x_\ell}{\partial q_j} n_j\right)$$

$$= m_\ell v_\ell^2 > 0 \qquad \text{where } v_\ell = \frac{\partial x_\ell}{\partial q_i} n_i$$

Expanding around $\bar{q}_0$

$$T = \frac{1}{2} m_{ij}\, \dot{q}_i \dot{q}_j = \frac{1}{2} m_{ij}\Big|_{q_0} \dot{q}_i \dot{q}_j + \dots$$

$$\simeq \frac{1}{2} T_{ij}\, \dot{q}_i \dot{q}_j = \frac{1}{2} T_{ij}\, \dot{\eta}_i \dot{\eta}_j$$

$$L \simeq \frac{1}{2}\left(T_{ij}\, \dot{\eta}_i \dot{\eta}_j - V_{ij}\, \eta_i \eta_j\right)$$

$$\frac{\partial L}{\partial \eta_i} = -V_{ij}\,\eta_j \qquad\qquad \frac{\partial L}{\partial \dot{\eta}_i} = T_{ij}\,\dot{\eta}_j$$

$$T_{ij}\,\ddot{\eta}_j = -V_{ij}\,\eta_j$$

Standard method for linear diff. eqs.

Independent solutions looked for in the form

$$\eta_j = a_j\, e^{-i\omega t}$$

General solution found from a superposition of those and their complex conjugates
(If η_i is a solution, then η_i^* too)

ω at this moment may be complex

Substituting, we get

$$-\omega^2 T_{ij}\, a_j = -V_{ij}\, a_j$$

or

$$\left(V_{ij} - \omega^2 T_{ij}\right) a_j = 0$$

$$\begin{vmatrix} V_{11}-\omega^2 T_{11} & V_{12}-\omega^2 T_{12} & \cdots \\ V_{21}-\omega^2 T_{21} & V_{22}-\omega^2 T_{22} & \\ V_{31}-\omega^2 T_{31} & & \end{vmatrix} = 0$$

Polynomial of nth degree in ω^2

↳ Generally n-roots ω^2.

The roots are real and nonnegative $\Rightarrow \omega$ - real

Proof:

$$a_i^* (V_{ij} - \omega^2 T_{ij}) a_j = 0$$

$$a_i^* V_{ij} a_j = \omega^2 a_i^* T_{ij} a_j$$

$$\omega^2 = \frac{a_i^* V_{ij} a_j}{a_i^* T_{ij} a_j} \qquad (\omega^2)^* = \frac{a_i V_{ij} a_j^*}{a_i T_{ij} a_j^*}$$

$$= \frac{a_j^* V_{ji} a_i}{a_j^* T_{ji} a_i}$$

$$\Rightarrow \omega^2 = (\omega^2)^*$$

Nonnegative, because T_{ij} & V_{ij} are positive for complex vectors just as well as for real vectors

$$a_i^* V_{ij} a_j = (\mathrm{Re}\, a_i - i\, \mathrm{Im}\, a_i) V_{ij} (\mathrm{Re}\, a_j + i\, \mathrm{Im}\, a_j)$$

$$= (\mathrm{Re}\, a_i) V_{ij} (\mathrm{Re}\, a_j) + (\mathrm{Im}\, a_i) V_{ij} (\mathrm{Im}\, a_j)$$

$$\underbrace{- i\, \mathrm{Im}\, a_i\, V_{ij}\, \mathrm{Re}\, a_j + i\, \mathrm{Re}\, a_i\, V_{ij}\, \mathrm{Im}\, a_j}_{=0} \geq 0$$

$a^*_i T_{ij} a_j > 0$

$\Rightarrow \omega^2 \geqslant 0$

ω_κ - normal frequencies of the system

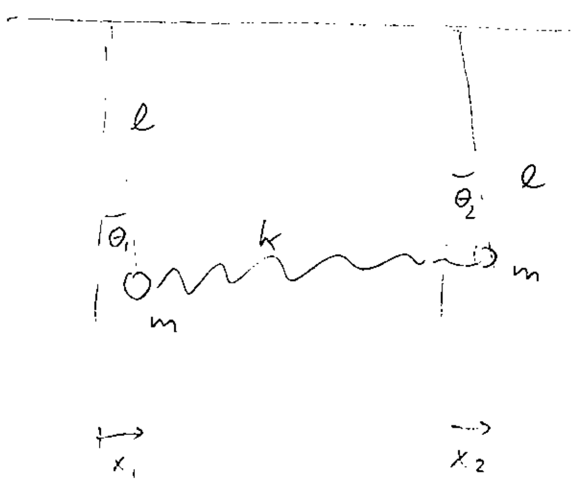
$(\bar{\bar{V}} - \omega_\lambda^2 \bar{\bar{T}}) \vec{a}_\lambda = 0$

solutions $\bar{a}_\lambda$ - amplitude vectors of normal modes of oscillation

may be normalized ...

Example:

Two identical simple pendulums are connected by a spring. Find a general solution to the equations of motion for small oscillations about equilibrium, within the plane of equilibrium positions. Assume that the natural length of the spring is equal to the separation between the equilibrium positions.



$T = \frac{m l^2 \dot{\theta}_1^2}{2}$

$V = -mgl\cos\theta$

for one pendulum

$\sin\theta_1 = \frac{x_1}{l}$

$\theta_1 \simeq \frac{x_1}{l}$ for small deviations

$T = \frac{m \dot{x}_1^2}{2}$ $\quad V = -mg\sqrt{l^2 - x_1^2}$

$= -mgl\sqrt{1 - \frac{x_1^2}{l^2}} \simeq -mgl + \frac{mg}{2l} x_1^2$

$$T = \frac{m\dot{x}_1^2}{2} + \frac{m\dot{x}_2^2}{2}$$

$$V = \frac{mg}{2\ell} x_1^2 + \frac{mg}{2\ell} x_2^2 + \frac{1}{2} k (x_1 - x_2)^2$$

$$= \frac{1}{2} \left\{ \left(\frac{mg}{\ell} + k\right) x_1^2 + \left(\frac{mg}{\ell} + k\right) x_2^2 - 2 k x_1 x_2 \right\}$$

$$\overline{\overline{T}} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \qquad \overline{\overline{V}} = \begin{pmatrix} \frac{mg}{\ell} + k & -k \\ -k & \frac{mg}{\ell} + k \end{pmatrix}$$

$$\begin{vmatrix} \frac{mg}{\ell} + k - m\omega^2 & -k \\ -k & \frac{mg}{\ell} + k - m\omega^2 \end{vmatrix} = 0$$

$$\left(\frac{mg}{\ell} + k - m\omega^2\right)^2 - k^2 = 0$$

$$m\omega^2 - \frac{mg}{\ell} - k = \pm k$$

$$\omega_1 = \sqrt{\frac{2k}{m} + \frac{g}{\ell}} \qquad \omega_2 = \sqrt{\frac{g}{\ell}}$$

first mode

$$\begin{pmatrix} -k & -k \\ -k & -k \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0 \quad \Rightarrow \quad a_1 = -a_2$$

$$\eta = C\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-i\omega_1 t} = |C| \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-i(\omega_1 t - \varphi)}$$

$$\tfrac{1}{2}(\eta + \eta^*) = |C| \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(\omega_1 t - \varphi_1)$$

$$\begin{pmatrix} k & -k \\ -k & k \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0 \quad \Rightarrow a_1 = a_2$$

$$\eta = C\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-i\omega_2 t} \qquad \tfrac{1}{2}(\eta + \eta^*) = C \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega_2 t - \varphi_2)$$

General solution

$$\vec{\eta} = C_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(\omega_1 t - \varphi_1) + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega_2 t - \varphi_2)$$

from now on real

$$x_1 = C_1 \cos(\omega_1 t - \varphi_1) + C_2 \cos(\omega_2 t - \varphi_2)$$

$$x_2 = -C_1 \cos(\omega_2 t - \varphi_1) + C_2 \cos(\omega_1 t - \varphi_2)$$

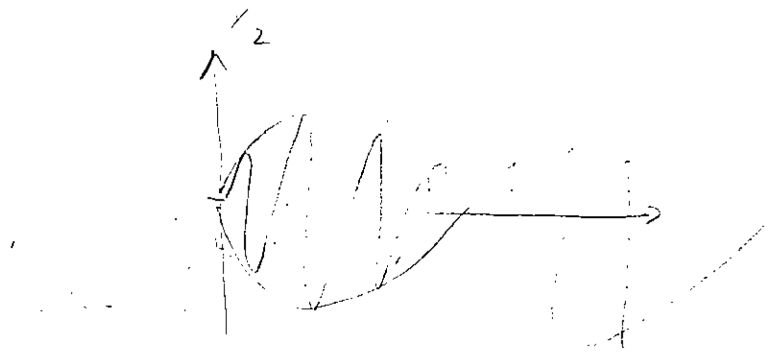
Example: Case of $\frac{k}{m} \ll \sqrt{\frac{g}{\ell}}$ weak spring $\qquad \omega_1 - \omega_2 \ll \omega_{1,2}$

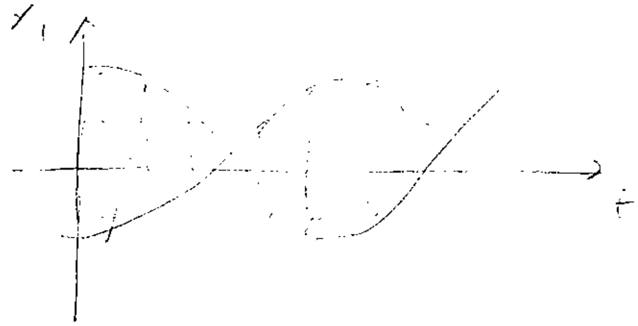
At $t=0$ $\quad x_2 = 0 \quad \varphi_1 = \varphi_2 = 0 \quad \Rightarrow C_1 = C_2 = C$

$$x_1 = C(\cos\omega_1 t + \cos\omega_2 t) = 2C\cos\left(\tfrac{\omega_1+\omega_2}{2}t\right)\cos\left(\tfrac{\omega_1-\omega_2}{2}t\right)$$

$$x_2 = C(\cos\omega_1 t - \cos\omega_2 t) = 2C\sin\left(\tfrac{\omega_1+\omega_2}{2}t\right)\sin\left(\tfrac{\omega_1-\omega_2}{2}t\right)$$

[illegible]





Beats

Normal coordinates

The eqs. for the amplitudes and frequencies of normal vibrations represent ~~the~~ a type of ~~an~~ the eigenvalue problem. In fact, the amplitudes for different frequencies satisfy a form of orthogonality relation:

$$\left(\overline{\overline{V}} - \omega_\lambda^2 \overline{\overline{T}}\right) \vec{a}_\lambda = 0$$

$$(V_{ij} - \omega_k^2 T_{ij}) a_{jk} = 0$$

$$(V_{ij} - \omega_\ell^2 T_{ij}) a_{j\ell} = 0$$

no summation over k or ℓ

$$a_{i\ell} V_{ij} a_{jk} - \omega_k^2 a_{i\ell} T_{ij} a_{jk} = 0$$

$$a_{ik} V_{ij} a_{j\ell} - \omega_\ell^2 a_{ik} T_{ij} a_{j\ell} = 0$$

Subtracting

$$(\omega_\ell^2 - \omega_k^2) a_{i\ell} T_{ij} a_{jk} = 0$$

If $\omega_\ell^2 \neq \omega_k^2$ $\qquad a_{i\ell} T_{ij} a_{jk} = 0$

The amplitudes may be modified by multiplying them by factors, and the normalization may be set by demanding

$$a_{i\ell} T_{ij} a_{j\ell} = 1$$

leading to the condition

$$a_{il} T_{ij} a_{jk} = \delta_{lk} \qquad \vec{a}_l \overline{\overline{T}} \vec{a}_k = \delta_{lk}$$

If any ω^2 is degenerate, the amplitude vectors within the respective subspace of ω^2 may be arranged to meet the condition by diagonalizing T within that subspace

Now, we have

$$\vec{a}_l (\overline{\overline{V}} - \omega_k^2 \overline{\overline{T}}) \vec{a}_k = 0$$

$$\vec{a}_l \overline{\overline{V}} \vec{a}_k - \omega_k^2 \vec{a}_l \overline{\overline{T}} \vec{a}_k = 0$$

$$\vec{a}_l \overline{\overline{V}} \vec{a}_k - \omega_k^2 \delta_{lk} = 0$$

$$\vec{a}_l \overline{\overline{V}} \vec{a}_k = \omega_k^2 \delta_{lk}$$

The set of normalized amplitude vectors diagonalizes the kinetic energy and potential matrices.

Introducing normal coordinates ζ_λ with
for normal mode λ

$$\eta_i = \sum_\lambda a_{i\lambda} \zeta_\lambda$$

$$L = \frac{1}{2} T_{ij} \dot{\eta}_i \dot{\eta}_j - \frac{1}{2} V_{ij} \eta_i \eta_j = \frac{1}{2} T_{ij} a_{i\lambda} a_{j\lambda'} \dot{\zeta}_\lambda \dot{\zeta}_{\lambda'}$$

$$- \frac{1}{2} V_{ij} a_{i\lambda} a_{j\lambda'} \zeta_\lambda \zeta_{\lambda'} = \frac{1}{2} \delta_{\lambda\lambda'} \dot{\zeta}_\lambda \dot{\zeta}_{\lambda'} - \frac{1}{2} \omega_\lambda^2 \delta_{\lambda\lambda'} \zeta_\lambda \zeta_{\lambda'}$$

$$= \sum_\lambda \frac{1}{2} (\dot{\zeta}_\lambda^2 - \omega_\lambda^2 \zeta_\lambda^2)$$

The eqs. for the normal coordinates are simply

$$\ddot{\zeta}_\lambda - \omega_\lambda^2 \zeta_\lambda = 0 \qquad \Rightarrow \qquad \zeta_\lambda = C_\lambda e^{-i\omega_\lambda t}$$

Normalized amplitudes for the two pendulums

$$\vec{a}_1 = N_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \vec{a}_2 = N_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\bar{\bar{T}} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$$

$$\vec{a}_1 \bar{\bar{T}} \vec{a}_1 = 1 \quad \Rightarrow \quad N_1^2 (m+m) = 1 \quad \Rightarrow N_1 = \frac{1}{\sqrt{2m}}$$

$$\vec{a}_2 \bar{\bar{T}} \vec{a}_2 = 1 \quad \Rightarrow \quad N_2^2 (m+m) = 1 \quad \Rightarrow N_2 = \frac{1}{\sqrt{2m}}$$

Normalized amplitudes for normal vibrations:

$$\vec{a}_1 = \begin{pmatrix} \frac{1}{\sqrt{2m}} \\ -\frac{1}{\sqrt{2m}} \end{pmatrix} \qquad \vec{a}_2 = \begin{pmatrix} \frac{1}{\sqrt{2m}} \\ \frac{1}{\sqrt{2m}} \end{pmatrix}$$

Note:

$$\omega_k^2 = 0$$

$$\ddot{\zeta}_k = 0 \quad \Rightarrow \quad \zeta_k(t) = \zeta_k(0) + \dot{\zeta}_k(0)\, t$$

This can be considered a limiting case of the harmonic solution

$$\zeta_k = A_k \cos(\omega_k t + \varphi_k) = A_k \cos\omega_k t \cos\varphi_k$$

$$+ A_k \sin\omega_k t \sin\varphi_k$$

$$\simeq \zeta_k(0) + \dot{\zeta}_k(0)\, t \qquad \text{for } t \ll \frac{1}{\omega_k}$$

as $\omega_k \to 0$ the range of the validity of the result expands.

When the potential is zero along a given coordinate and mass is constant, this indeed becomes a valid solution as the displacement grows.

Vanishing normal frequencies are found for systems of particles for coordinates describing the translational motion of the center of mass and for rotations about the center of mass.

Forced vibrations

If a system is forced to vibrate around equilibrium under an influence of external periodic forces, then, after a long time the system oscillates at the frequency of the forces, rather than the resonance frequencies.

$\eta_i = a_{ik} \zeta_k$ amplitude vec.

Generalized forces:

$$Q_k = F_j \frac{\partial \eta_j}{\partial \zeta_k} = F_j a_{jk}$$

$$\ddot{\zeta}_k + \omega_k^2 \zeta_k = Q_k$$

$$Q_k = Q_{k0} \cos(\omega t + \delta_k)$$

$$\zeta_k = B_k \cos(\omega t + \varphi_k)$$

$$-\omega^2 \zeta_k + \omega_k^2 \zeta_k = Q_k$$

$$\zeta_k = \frac{Q_k}{\omega_k^2 - \omega^2} = \frac{Q_{k0}}{\omega_k^2 - \omega^2} \cos(\omega t + \delta_k)$$

$$\eta_i = \frac{a_{ik} Q_{k0} \cos(\omega t + \delta_k)}{\omega_k^2 - \omega^2}$$

If ω approaches any of the resonant frequencies, the amplitude increases indiscriminately.

The behavior changes when including the effects of dissipation

Forces proportional to velocity may be represented in terms of the Rayleigh dissipation function

$$\mathcal{F} = \frac{1}{2} K_{ij} \dot{x}_i \dot{x}_j$$

where $\bar{\bar{K}} \geq 0$ and the friction forces are

$$F_i = -\frac{\partial \mathcal{F}}{\partial \dot{x}_i} = -K_{ij} \dot{x}_j$$

In terms of displacement coordinates η

$$\mathcal{F} = \frac{1}{2} \mathcal{F}_{ij} \dot{\eta}_i \dot{\eta}_j \qquad \mathcal{F}_{ij} = \frac{\partial x_k}{\partial \eta_i} \frac{\partial x_l}{\partial \eta_j} K_{kl}$$

With friction forces but no driving forces

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\eta}_i} - \frac{\partial L}{\partial \eta_i} = Q_i = -\frac{\partial \mathcal{F}}{\partial \dot{\eta}_i}$$

$$T_{ij} \ddot{\eta}_j + V_{ij} \eta_j = -\mathcal{F}_{ij} \dot{\eta}_j$$

The solution is complicated by the fact that 3 different matrices cannot be normally simultaneously diagonalized by a coordinate transformation.

For the moment let us assume that a diagonalization is possible. Then

$$\ddot{\zeta}_i + \mathcal{F}_i \dot{\zeta}_i + w_i^2 \zeta_i = 0 \qquad \zeta_i = C_i e^{\gamma t}$$

$$\gamma^2 + \mathcal{F}_i \gamma + w_i^2 = 0$$

$$\Delta = \mathcal{F}_i^2 - 4 w_i^2$$

For $\mathcal{F}_i \ll \omega_i$ ← small damping

$$\gamma = \frac{-\mathcal{F}_i \pm \sqrt{\mathcal{F}_i^2 - 4\omega_i^2}}{2}$$

$$= -\frac{\mathcal{F}_i}{2} \pm i\sqrt{\omega_i^2 - \frac{\mathcal{F}_i^2}{4}} \approx -\frac{\mathcal{F}_i}{2} \pm i\omega_i$$

$$\zeta_i = C_i e^{-i\omega_i t} e^{-\frac{\mathcal{F}_i}{2} t}$$

$$\zeta_i = |C_i| e^{-\mathcal{F}_i/2\, t} \cos(\omega_i t + \phi_i)$$

amplitude decreases with time

If the matrices cannot be simultaneously diagonalized, we can still look for solutions in the form

$$\eta_i = C a_i e^{\gamma t}$$

This yields

$$(\bar{\bar{V}} + \gamma \bar{\bar{\mathcal{F}}} + \gamma^2 \bar{\bar{T}})\vec{a} = 0$$

Condition

$$|\bar{\bar{V}} + \gamma \bar{\bar{\mathcal{F}}} + \gamma^2 \bar{\bar{T}}| = 0$$

$2n$-degree polynomial in γ → $2n$ roots

If γ is a root then γ^* is also a root

Real part of γ is nonpositive:

$$\vec{a}^* \bar{\bar{V}} \vec{a} + \gamma \, \vec{a}^* \bar{\bar{F}} \vec{a} + \gamma^2 \vec{a}^* \bar{\bar{T}} \vec{a} = 0$$

conj $$\vec{a}^* \bar{\bar{V}} \vec{a} + \gamma^* \vec{a}^* \bar{\bar{F}} \vec{a} + \gamma^{*2} \vec{a}^* \bar{\bar{T}} \vec{a} = 0$$

subtract

$$(\gamma - \gamma^*) \, \vec{a}^* \bar{\bar{F}} \vec{a} + (\gamma - \gamma^*)(\gamma + \gamma^*) \, \vec{a}^* \bar{\bar{T}} \vec{a} = 0$$

$$\Rightarrow \quad \gamma + \gamma^* = - \frac{\vec{a}^* \bar{\bar{F}} \vec{a}}{\vec{a}^* \bar{\bar{T}} \vec{a}} \leq 0$$

$$\gamma_k = -i\,\omega_k = -i\left(\omega_{0k} - i\,\frac{\Gamma_k}{2}\right)$$

$k = 1, \dots, n$ real

$$\Gamma_k = +\frac{1}{4}\,\frac{\vec{a}_k^{\,*}\,\bar{\bar{\mathcal{F}}}\,\vec{a}_k}{\vec{a}_k^{\,*}\,\bar{\bar{T}}\,a_k}$$

other solutions γ_k^*

$$\text{Re}\,\gamma_k \le 0 \iff \Gamma_k \ge 0$$

γ_k^* also solution $\iff$ $i\omega_k^* = -i\left(-\omega_{0k} - i\,\frac{\Gamma_k}{2}\right)$ also a solution

Oscillating driving forces $\quad F_j = F_{0j}\,e^{-i\omega t}$ (F_{0j} complex)

EOM

$$T_{ij}\,\ddot{\eta}_j + V_{ij}\,\eta_j = \underbrace{F_{0j}\,e^{-i\omega t}}_{\text{driving}} - \underbrace{\mathcal{F}_{ij}\,\dot{\eta}_j}_{\text{friction}}$$

$$\eta_j = a_j\,e^{-i\omega t}$$

$$\underbrace{(-\omega^2 T_{ij} + V_{ij} - i\omega\,\mathcal{F}_{ij})}_{M_{ij}(\omega)}\,a_j = F_{0j}$$

Nonhomogenous eq. $\qquad \bar{\bar{M}}(\omega)\,\vec{a} = \vec{F}$

Solution $\qquad a_i = \dfrac{|\bar{\bar{M}}^{(i)}(\omega)|}{|\bar{\bar{M}}(\omega)|}$

$$\begin{array}{c} \\ \end{array}\begin{vmatrix} M_{11} & M_{12} & F_1 & M_{1\,i+1} & \cdots & M_{1n} \\ \vdots & & \vdots & & & \vdots \\ M_{n1} & M_{n2} & F_n & M_{n\,i+1} & & M_{nn} \end{vmatrix}$$

$\hat{i}$ ith column replaced by $\vec{F}$

$|\bar{\bar{M}}(\omega)|$ -- 2n degree polynomial

such as in the problem of free variations

roots are the ω_k-values, $k = 1, .. n$, and $-\omega_k^*$.

This means that:

$$|\bar{\bar{M}}(\omega)| = G\,(\omega - \omega_1)\dots(\omega - \omega_k)(\omega + \omega_1^*)\dots(\omega + \omega_k^*)$$

$$= G \prod_{i=1}^{n} (\omega - \omega_{0k} + i\Gamma_k/2)(\omega + \omega_{0k} + i\Gamma_k/2)$$

$$\eta_j = \frac{-|\bar{\bar{M}}^{(j)}(\omega)|}{|\bar{\bar{M}}(\omega)|}\, e^{-i\omega t}$$

Real solution

$$\frac{1}{2}(\eta_j + \eta_j^*) = \frac{1}{2}\left(\frac{|\bar{\bar{M}}^{(j)}(\omega)|}{|\bar{\bar{M}}(\omega)|}\, e^{-i\omega t} + \frac{|\bar{\bar{M}}^{(j)*}(\omega)|}{|\bar{\bar{M}}^*(\omega)|}\, e^{i\omega t}\right)$$

$$= \frac{\dots}{|\bar{\bar{M}}(\omega)|\,|\bar{\bar{M}}^*(\omega)|} \cos(\omega t + \varphi_j)$$

$$|\bar{\bar{M}}(\omega)|\,|\bar{\bar{M}}^*(\omega)| = |G|^2 \prod_{i=1}^{n} \left((\omega - \omega_{0k})^2 + \Gamma_k^2/4\right)\left((\omega + \omega_{0k})^2 + \Gamma_k^2/4\right)$$

The amplitudes grow again as ω approaches any of the resonance frequencies but the growth is now moderated by dissipation represented by Γ.