1. Carroll & Ostlie: problem 5.14

A white dwarf is a very dense star, with it ions and electrons packed extremely closely together. Each electron may be considered to be located within a region of size $\Delta x \approx 1.5 \times 10^{-10}$ cm. Use Heisenberg’s uncertainty principle, Eq. (5.18), to estimate the minimum speed of the electron. Do you think that the effects of relativity will be important for these stars?

$$\Delta v = \frac{\Delta p}{m} \approx \frac{\hbar}{m_c \Delta x}$$

$$\Delta v = \frac{6.62 \times 10^{-27}}{2\pi 9.1 \times 10^{-28} 1.5 \times 10^{-10}} = 7.7 \times 10^9 \text{cm/s}$$

Relativistic corrections go as $(v/c)^2$, and here $v/c=0.26$, so relativistic corrections are about 6%. Note that the thermal speed of electrons at the center of the Sun is slightly less, $\approx 2 \times 10^9$ cm/s.

2. Carroll & Ostlie: problem 5.15

An electron spends roughly $10^{-8}$ s in the first excited stat of the hydrogen atom before making a spontaneous downward transition to the ground state.

(a) Use Heisenberg’s uncertainty principle (Eq. 5.19) to determine the uncertainty $\Delta E$ in the energy of the first excited state.

$$\Delta E \approx \frac{\hbar}{(2\pi \Delta t)}$$

The lifetime of the level is the uncertainty in time, so the uncertainty in energy of level 2 is

$$\Delta E = 1.05 \times 10^{-19} \text{erg} = 6.58 \times 10^{-8} \text{eV}$$

(b) Calculate the uncertainty $\Delta \lambda$ in the wavelength of the photon involved in a transition (either upward or downward) between the ground and first excited states of the hydrogen atom. Why can you assume that $\Delta E = 0$ for the ground state?

$$\Delta \nu \approx \frac{\Delta E}{h} = \frac{c}{\lambda^2} \Delta \lambda.$$
The wavelength of $\lambda_{\alpha} = 1.215 \times 10^{-5} \text{ cm}$, so the uncertainty in the wavelength is

$$
\Delta \lambda \approx \lambda^2 \Delta E \frac{\lambda}{hc} = \left(1.215 \times 10^{-5}\right)^2 \frac{(1.05 \times 10^{-19})}{(6.62 \times 10^{-34})(3 \times 10^{10})}
$$

$$
\Delta \lambda \approx 7.8 \times 10^{-14} \text{ cm} = 7.8 \times 10^{-7} \text{ nm}
$$

which is much less than the thermal Doppler width.

3. Carroll & Ostlie: problem 11.5a

(a) Using Eq. (9.58) and neglecting turbulence, estimate the full width at half-maximum of the hydrogen $H_{\alpha}$ absorption line due to random thermal motions in the Sun’s photosphere. Assume that the temperature is the Sun’s effective temperature.

Eq 9.58 gives the full width at half-maximum

$$
\Delta \lambda_{1/2} = \frac{2\lambda}{c} \left[ \left( \frac{2kT}{m} + v_{\text{turb}}^2 \right) \ln 2 \right]^{1/2}
$$

Here,

$$
\begin{align*}
\lambda &= 656.2 \text{ nm} = 6.562 \times 10^{-5} \text{ cm} \quad (1) \\
c &= 2.998 \times 10^{10} \text{ cm/s} \quad (2) \\
k &= 1.38 \times 10^{-24} \text{ erg/K} \quad (3) \\
m &= 1.67 \times 10^{-24} \text{ g} \quad (4) \\
T_{\text{eff}} &= 5770 \text{ K} \quad (5) \\

\end{align*}
$$

so

$$
\Delta \lambda_{1/2} = 3.56 \times 10^{-9} \text{ cm} = 0.0356 \text{ nm}
$$

4. Carroll & Ostlie: problem 9.11a

According to the “standard model” of the Sun, the central density is $162 \text{ g cm}^{-3}$ and the Rosseland mean opacity at the center is $1.16 \text{ cm}^2 \text{ g}^{-1}$.

(a) Calculate the mean free path of a photon at the center of the Sun.

$$
\ell = 1/n\sigma = 1/\rho\kappa = 1/(162 \text{ g cm}^{-3} \times 1.16 \text{ cm}^2 \text{ g}^{-1}) = 5.32 \times 10^{-3} \text{ cm}.
$$
5. Carroll & Ostlie: problem 7.4

Sirius is a visual binary with a period of 49.94 yr. Its measured trigonometric parallax is $0.377''$ and, assuming that the plane of the orbit is in the plane of the sky, the true angular extent of the semimajor axis of the reduced mass is $\alpha = 7.62''$. The ratio of the distances of Sirius A and Sirius B from the center of mass is $a_A/a_B = 0.466$.

(a) Find the mass of each member of the system.

The distance to Sirius is

$$D = 1/\pi'' = 2.65pc = 8.178 \times 10^{18} \text{ cm}.$$  

The semimajor axis is

$$a = \alpha(\text{rad})D = 7.62'' \times \frac{2\pi \text{ rad}}{1.296 \times 10^6''} \times 8.178 \times 10^{18} = 3.02 \times 10^{14} \text{ cm}.$$  

Then the sum of the masses is

$$M_A + M_B = \frac{4\pi^2 a^3}{G P^2} = \frac{4\pi^2}{6.67 \times 10^{-8} (49.49 \times 3.156 \times 10^7)^2} \left(\frac{3.02 \times 10^{14}}{49.49 \times 3.156 \times 10^7}\right)^3 = 6.68 \times 10^{33} \text{ g} = 3.36 \text{ M}_\odot.$$  

The ratio of masses is

$$M_A/M_B = M_B a_B/M_A a_A = a_A/a_B$$

so

$$M_A (1 + a_A/a_B) = 3.36 \text{ M}_\odot$$

or

$$M_A = 3.36/(1 + 0.466) = 2.29 \text{ M}_\odot,$$

and

$$M_B = 3.36 - 2.29 = 1.07 \text{ M}_\odot.$$  

(b) The absolute bolometric magnitude of Sirius A is 1.33 and Sirius B has a absolute bolometric magnitude of 8.57. Determine their luminosities, in units of the solar luminosity.

The magnitude is related to the flux by

$$m_1 - m_2 = -2.5 \log_{10} \left(\frac{F_1}{F_2}\right)$$

$$\frac{F_1}{F_2} = 10^{-(m_1 - m_2)/2.5}.$$  

The flux for a star at distance D and luminosity L is

$$F = L/4\pi D^2.$$
For stars at the same distance the ratio of luminosities = the ratio of fluxes. For absolute magnitudes the distance is the nominal distance of 10 pc, and the absolute bolometric magnitude of the Sun is $M_{b,\odot} = 4.76$, so

$$\frac{L_A}{L_\odot} = 10^{(4.76-1.33)/2.5} = 10^{1.37} = 23.6 \quad (9)$$

$$\frac{L_B}{L_\odot} = 10^{(4.76-8.57)/2.5} = 10^{-1.52} = 0.03 \quad (10)$$

(c) The effective temperature of Sirius B is estimated to be approximately 27,000 K. Estimate its radius and compare it to the radii of the Sun and Earth.

The luminosity for given radius and effective temperature is

$$L = 4\pi R^2 \sigma T_{\text{eff}}^4$$

so

$$R = \left( \frac{L}{4\pi \sigma T_{\text{eff}}^4} \right)^{1/2},$$

or

$$R_B = \left( \frac{0.03 \times 3.83 \times 10^{33}}{4\pi \times 5.67 \times 10^{-5} \times (2.7 \times 10^4)^4} \right)^{1/2} = 5.5 \times 10^8 \text{ cm} = 8 \times 10^{-3} R_\odot = 0.86 R_\oplus.$$