Astronomy 304, STARS

Homework # 5, Due Wednesday, Feb. 12, 2003

Show All Your Work

1. Calculate the pressure exerted by a gas of non-relativistic degenerate neutrons in terms of the mass density. Get a numerical expression with density in units of g cm^{-3}. What density would be needed to make this pressure of the same order of magnitude as our estimate for the central pressure of the Sun? How does this compare with the current central density of the Sun of about 160 g cm^{-3}?

   As done in class the expression for the pressure of a non-relativistic degenerate gas is

$$P = \frac{1}{3}npv \quad (1)$$

$$p = \hbar/\Delta x \quad (2)$$

$$\Delta x = n^{-1/3} \quad (3)$$

$$p = \hbar n^{1/3} \quad (4)$$

$$v = p/m \quad (5)$$

$$P = \frac{1}{3}\frac{\hbar^2 n^{5/3}}{m} \quad (6)$$

   Here n is the number density of neutrons and m is the neutron mass. Express this in terms of the mass density. For a neutron star, almost all the nucleons are neutrons, so $\rho = n_{\text{neutron}} m_{\text{neutron}}$. Then,

$$P = \frac{1}{3}\frac{\hbar^2 \rho^{5/3}}{m^{8/3}}. \quad (7)$$

   Now $\hbar = 1.055 \times 10^{-27}$ erg s,
   and $m_{\text{neutron}} = 1.675 \times 10^{-24}$ gm,
   then

$$P = 9.38 \times 10^8 \rho^{5/3} \quad (8)$$

   To match the pressure in the center of the Sun, which is 2.5×10^{17} dynes cm^{-2}, requires a density of

$$\rho = \left(\frac{2.5 \times 10^{17}}{9.38 \times 10^8}\right)^{3/5} = 1.1 \times 10^5 \text{ g/cm}^3 \quad (9)$$

   In class we made an estimate of the central pressure of the Sun as 1×10^{15} dynes cm^{-2}, requires a density of

$$\rho = \left(\frac{1 \times 10^{15}}{9.38 \times 10^8}\right)^{3/5} = 4 \times 10^3 \text{ g/cm}^3 \quad (10)$$

2. Derive the formula for the pressure of a gas of degenerate extremely relativistic particles (you need not get the constants exactly correct),

$$P = \frac{1}{8}\left(\frac{3}{\pi}\right)^{1/3} hcn^{4/3}$$

Since an expression for the pressure is what is wanted, start from the general expression for the pressure due to collisions,

$$P = \frac{1}{3} n(mv)v \ .$$

For a degenerate gas the momentum mv is determined by the Heisenberg Uncertainty Principle, since that is what is meant by a particle being degenerate,

$$mv = p = \frac{\hbar}{\Delta x} \ .$$

The uncertainity in position is the space between the particles,

$$\Delta x = n^{-1/3} \ .$$

For extremely relativistic particles, their speed is the speed of light, c. Hence, substituting these values into the expression for the pressure,

$$P = \frac{1}{3} \hbar n^{4/3} c \ . \tag{11}$$

This has the correct dependence on the dimensional quantities, h c and n, but does not get the constants exactly right.

3. Carroll & Ostlie: problem 10.9

Calculate the ratio of the energy generation rate for the pp chain to the energy generation rate for the CNO cycle give conditions characteristic of the center of the present day (evolved) Sun, namely T = 1.58×10^7 K, $\rho = 162$ g/cm^3, $X = 0.34$, and $X_{CNO} = 0.013$. Assume that the pp chain screening factor is unity ($f_{pp} = 1$) and that the pp chain branching factor is unity ($\psi_{pp} = 1$). For the pp chain,

$$\epsilon_{pp} = 2.38 \times 10^6 \rho X^2 f_{pp} \psi_{pp} C_{pp} T_6^{-2/3} e - 33.80 T_6^{-1/3} \, \mathrm{erg\, g^{-1}\, s^{-1}}$$

or, near $T = 1.5 \times 10^7$ K,

$$\epsilon_{pp} = 1.07 \times 10^{-5} \rho X^2 \psi_{pp} f_{pp} T_6^4 \, \mathrm{erg\, g^{-1}\, s^{-1}}$$

Under solar conditions,

$$\epsilon_{pp} = 1.07 \times 10^{-5} \times 162 \times 0.34^2 \times 15.8^4 = 12.5 \, \mathrm{erg\, g^{-1}\, s^{-1}} \ .$$

For the CNO cycle,

$$\epsilon_{CNO} = 8.67 \times 10^{27} \rho X X_{CNO} C_{CNO} T_6^{-2/3} e^{-152.28 T_6^{-1/3}} \, \mathrm{erg\, g^{-1}\, s^{-1}}$$

or, near $T = 1.5 \times 10^7$ K,

$$\epsilon_{CNO} = 8.24 \times 10^{-24} \rho X X_{CNO} T_6^{19.9} \, \mathrm{erg\, g^{-1}\, s^{-1}} \ .$$

Under solar conditions,

$$\epsilon_{CNO} = 8.24 \times 10^{-24} \times 162 \times 0.34 \times 0.013 \times 15.8^{19.9} = 4.21 \, \mathrm{erg\, g^{-1}\, s^{-1}} \ .$$

4. Carroll & Ostlie: problem 10.12 (a) and (b)

Calculate the amount of energy released or absorbed in the following reactions (express your answers in MeV):

$$\begin{aligned}
(a)\, {}^{12}_{6}C + {}^{12}_{6}C &\rightarrow {}^{24}_{12}Mg + \gamma && (12)\\
(b)\, {}^{12}_{6}C + {}^{12}_{6}C &\rightarrow {}^{16}_{8}O + 2\,{}^{4}_{2}He && (13)\\
& && (14)
\end{aligned}$$

The mass of ${}^{12}_{6}C$ is 12.0000 u, by definition, and the masses of ${}^{16}_{8}O$, ${}^{24}_{12}Mg$ are 15.99491 u and 23.98504 u, respectively. Are these reactions exothermic or endothermic?

$$\begin{aligned}
Q &= \Delta mc^2 = \Delta m(u)c^2\, 931\,\mathrm{MeV/c^2} = \Delta m(u)\, 931\,\mathrm{MeV} && (15)\\
\Delta m &= \sum m_{\mathrm{initial}}(u) - \sum m_{\mathrm{final}}(u) && (16)\\
(a)\, Q &= 24 - 23.98504 = 0.015u = 0.015 \times 931.49432 && (17)\\
&= 13.9 MeV && (18)\\
(b)\, Q &= 24 - 15.99491 - 2 \times 4.002603 = -1.16 \times 10^{-4} u && (19)\\
&= -1.16 \times 10^{-4} \times 931.49432 = -0.108 MeV && (20)
\end{aligned}$$

5. Carroll & Ostlie: problem 10.15

Estimate the hydrogen burning lifetimes of stars on the lower and upper ends of the main sequence. The lower end of the main sequence occurs near 0.085 $M_\odot$, with $\log_{10} T_e = 3.438$ and $\log_{10} (L/L_\odot) = -3.297$, while the upper end of the main sequence occurs at approximately 90 $M_\odot$ with $\log_{10} T_e = 4.722$ and $\log_{10} (L/L_\odot) = 6.045$. Assume that the 0.085 $M_\odot$ star is entirely convective so that, through convective mixing, all of its hydrogen becomes available for burning rather than just the inner 10%.

$$t_{\mathrm{MS}} = \frac{E_{\mathrm{hydrogen}}}{L}$$

and

$$E_{\mathrm{hydrogen}} = 0.007 f X M c^2 \;,$$

where f = fraction of hydrogen fused into helium and X = mass fraction of hydrogen ≈ 0.7.

$$\begin{aligned}
t_{\mathrm{upper}} &= 0.007 \times 0.1 \times 0.7 \times 90 \times 1.989 \times 10^{33} \times 8.99 \times 10^{20} && (21)\\
& \quad /1.11 \times 10^{6} \times 3.826 \times 10^{33} && (22)\\
&= 5.9 \times 10^{6} \mathrm{years} && (23)\\
t_{\mathrm{lower}} &= 0.007 \times \times 0.7 \times 0.085 \times 1.989 \times 10^{33} \times 8.99 \times 10^{20} && (24)\\
& \quad /5.05 \times 10^{-4} \times 3.826 \times 10^{33} && (25)\\
&= 1.2 \times 10^{13} \mathrm{years} && (26)
\end{aligned}$$

6. Carroll & Ostlie: problem 10.16

Using the information given in problem 10.15, calculate the radii of a 0.085 $M_\odot$ star and a 90 $M_\odot$ star. What is the ratio of their radii?

$$L = 4\pi R^2 \sigma T_e^4 \; ,$$

so

$$\frac{R}{R_\odot} = \left(\frac{L}{L_\odot}\right)^{1/2} \left(\frac{T_{e\odot}}{T_e}\right)^2$$

$$\left(\frac{R}{R_\odot}\right)_{\mathrm{upper}} = 1.05 \times 10^3/83.5 = 12.6 \qquad (27)$$

$$\left(\frac{R}{R_\odot}\right)_{\mathrm{lower}} = 2.25 \times 10^{-2}/0.226 = 0.1 \qquad (28)$$

The ratio of their radii is 126, much less than the ratio of their masses.