Physics 410 - 2003

Thermal Physics

Final Exam

1. For a single-particle state s of an ideal Fermi gas, find fluctuations in the average occupancy. This means that you have to calculate $\langle(\Delta N_s)^2\rangle \equiv \langle(N_s - \langle N_s\rangle)^2\rangle$, where N_s is the number of particles and $\langle\ldots\rangle$ means statistical averaging. Express $\langle(\Delta N_s)^2\rangle$ in terms of $\langle N_s\rangle$ (10 pt). Plot (schematically) $\langle(\Delta N_s)^2\rangle$ as a function of $(\varepsilon_s - \mu)/\tau$ for $\mu/\tau \gg 1$, where ε_s is the state energy and μ is the chemical potential (5 pt).

2. In an ultra-relativistic ideal monatomic gas, the energy of a particle ε is related to its momentum $\mathbf{p}$ by the expression $\varepsilon = c|\mathbf{p}|$. Find the interrelation between pressure p and energy density of the gas U/V (do not confuse pressure p with the particle momentum $\mathbf{p}$) (10 pt)

3. Using Maxwell's relations, show that

$$\left(\frac{\partial C_V}{\partial V}\right)_\tau = \tau\left(\frac{\partial^2 p}{\partial \tau^2}\right)_V .$$

   Show that this equation applies to an ideal gas (you can use results for an ideal gas without derivation; the gas does not have to be monatomic) (10 pt)

4. Consider a photon gas in a very thin cavity, so that this gas may be supposed to be two-dimensional. Assume that electromagnetic waves in the cavity have only one polarization and that the area of the cavity is A. Find the energy of photons at temperature τ (10 pt)

You need to have 30 points.

Some useful expressions:

$$\int_0^\infty \frac{x^n\,dx}{\exp(x)-1} = n!\zeta(n+1), \quad \zeta(2) \approx 1.64;\ \zeta(3) \approx 1.20;\ \zeta(4) \approx 1.08$$

Good luck!

Solutions

Problem 1. The probability for a state s to be occupied by N particles ($N = 0, 1$) is

$$P_s(N) = \mathcal{Z}_s^{-1} \exp[N(\mu - \varepsilon_s)/\tau], \quad \mathcal{Z}_s = 1 + \exp[(\mu - \varepsilon_s)/\tau]$$

This gives

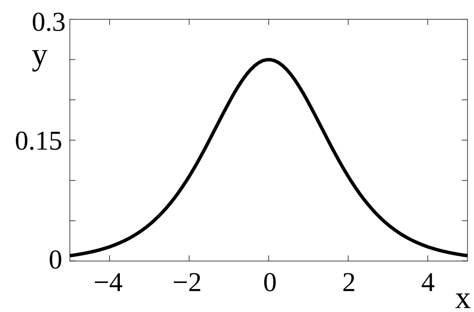
$$\langle N_s \rangle = \sum_N N P_s(N) = \frac{1}{\exp[(\varepsilon_s - \mu)/\tau] + 1}$$

$$\langle N_s^2 \rangle = \sum_N N^2 P_s(N) = \frac{1}{\exp[(\varepsilon_s - \mu)/\tau] + 1} \equiv \langle N_s \rangle$$

Therefore $\langle (N_s - \langle N_s \rangle)^2 \rangle \equiv \langle N_s^2 \rangle - \langle N_s \rangle^2 = \langle N_s \rangle (1 - \langle N_s \rangle)$.
This can be written as

$$y \equiv \langle (N_s - \langle N_s \rangle)^2 \rangle = \frac{1}{\exp(x) + 1} \times \frac{1}{\exp(-x) + 1}$$

where $x = (\varepsilon_s - \mu)/\tau$. For large μ/τ the function y sharply peaks at $\varepsilon_s = \mu$, as shown in the sketch.



Problem 2. The most straightforward way of solving this problem is based on calculating the partition function Z. For an ideal gas of N atoms $Z = Z_1^N / N!$, where

$$Z_1 = \int \frac{d^3p\, d^3q}{(2\pi\hbar)^3}\, e^{-\varepsilon(|\mathbf{p}|)/\tau} = \frac{V}{(2\pi\hbar)^3} \int d^3p\, e^{-cp/\tau}.$$

This gives

$$Z_1 = \frac{V}{(2\pi\hbar)^3} \times 4\pi \int_0^\infty p^2 dp\, e^{-cp/\tau} = \frac{V\tau^3}{\pi^2 (\hbar c)^3}$$

We know that pressure is

$$p = -\frac{\partial F}{\partial V} = \tau \frac{\partial \log Z}{\partial V} = \frac{N\tau}{V}.$$

The internal energy is

$$U = \tau^2 \frac{\partial \log Z}{\partial \tau} = 3N\tau.$$

Therefore we obtain $p = U/3V$.

Problem 3. From $dF = -\sigma\, d\tau - p\, dV$ we have $\sigma = -(\partial F/\partial \tau)_V$ and $p = -(\partial F/\partial V)_\tau$. Then from $C_V = \tau(\partial \sigma/\partial \tau)_V$ we have

$$\left(\frac{\partial C_V}{\partial V} \right)_\tau = \tau \frac{\partial^3 F}{\partial V \partial \tau^2} \equiv \tau \left(\frac{\partial^2 p}{\partial \tau^2} \right)_V.$$

For an ideal gas, the internal energy is a sum of energies of individual molecules, and therefore it is independent of V for a given number of molecules. As a consequence, $\partial C_V/\partial V = 0$. On the other hand, pressure is determined only by translational motion of molecules, and therefore $p = N\tau/V$, from which we have $\partial^2 p/\partial \tau^2 = 0$, i.e. the relation between the derivatives of C_V and p is satisfied.

Problem 4. The occupation of a mode with frequency ω is $\langle s(\omega)\rangle = [\exp(\hbar\omega/\tau) - 1]^{-1}$. We will model a 2D system by a square with side L, as we did before, $A = L^2$. The frequency of the mode with the quantum number $\mathbf{n} = (n_x, n_y)$ is $\omega_{\mathbf{n}} = |\mathbf{n}|\pi c/L$. The sum over the mode quantum numbers is

$$\sum_{\mathbf{n}}(\ldots) = \frac{1}{4}\int_0^\infty 2\pi n\, dn(\ldots)$$

Therefore the internal energy

$$U = \sum_{\mathbf{n}} \hbar\omega_{\mathbf{n}}\langle s(\omega_{\mathbf{n}})\rangle = \frac{\hbar L^2}{2\pi c^2}\int_0^\infty \omega^2\langle s(\omega)\rangle d\omega = \frac{L^2\tau^3}{2\pi\hbar^2 c^2}\int_0^\infty \frac{x^2}{\exp(x)-1}dx = \frac{A}{\pi\hbar^2 c^2}\tau^3\zeta(3)$$

where we changed from integration over n to integration over $\omega = Ln/\pi c$ and then over $x = \hbar\omega/\tau$.