1. For a single-particle state $s$ of an ideal Fermi gas, find fluctuations in the average occupancy. This means that you have to calculate $\langle (\Delta N_s)^2 \rangle \equiv \langle (N_s - \langle N_s \rangle)^2 \rangle$, where $N_s$ is the number of particles and $\langle \ldots \rangle$ means statistical averaging. Express $\langle (\Delta N_s)^2 \rangle$ in terms of $\langle N_s \rangle$ (10 pt). Plot (schematically) $\langle (\Delta N_s)^2 \rangle$ as a function of $(\varepsilon_s - \mu)/\tau$ for $\mu/\tau \gg 1$, where $\varepsilon_s$ is the state energy and $\mu$ is the chemical potential (5 pt).

2. In an ultra-relativistic ideal monatomic gas, the energy of a particle $\varepsilon$ is related to its momentum $p$ by the expression $\varepsilon = c|p|$. Find the interrelation between pressure $p$ and energy density of the gas $U/V$ (do not confuse pressure $p$ with the particle momentum $p$) (10 pt).

3. Using Maxwell’s relations, show that

$$\left( \frac{\partial C_V}{\partial V} \right)_T = \tau \left( \frac{\partial^2 p}{\partial \tau^2} \right)_V.$$

Show that this equation applies to an ideal gas (you can use results for an ideal gas without derivation; the gas does not have to be monatomic) (10 pt).

4. Consider a photon gas in a very thin cavity, so that this gas may be supposed to be two-dimensional. Assume that electromagnetic waves in the cavity have only one polarization and that the area of the cavity is $A$. Find the energy of photons at temperature $\tau$ (10 pt).

You need to have 30 points.

Some useful expressions:

$$\int_0^\infty \frac{x^n \, dx}{\exp(x) - 1} = n! \zeta(n + 1), \quad \zeta(2) \approx 1.64; \quad \zeta(3) \approx 1.20; \quad \zeta(4) \approx 1.08$$

Good luck!
Solutions

Problem 1. The probability for a state $s$ to be occupied by $N$ particles ($N = 0, 1$) is

$$P_s(N) = Z_s^{-1} \exp[N(\mu - \varepsilon_s)/\tau], \quad Z_s = 1 + \exp[(\mu - \varepsilon_s)/\tau]$$

This gives

$$\langle N_s \rangle = \sum_N N P_s(N) = \frac{1}{\exp[(\varepsilon_s - \mu)/\tau] + 1}$$

$$\langle N_s^2 \rangle = \sum_N N^2 P_s(N) = \frac{1}{\exp[(\varepsilon_s - \mu)/\tau] + 1} \equiv \langle N_s \rangle$$

Therefore $\langle (N_s - \langle N_s \rangle)^2 \rangle = \langle N_s^2 \rangle - \langle N_s \rangle^2 = \langle N_s \rangle (1 - \langle N_s \rangle)$.

This can be written as

$$y \equiv \langle (N_s - \langle N_s \rangle)^2 \rangle = \frac{1}{\exp(x) + 1} \times \frac{1}{\exp(-x) + 1}$$

where $x = (\varepsilon_s - \mu)/\tau$. For large $\mu/\tau$ the function $y$ sharply peaks at $\varepsilon_s = \mu$, as shown in the sketch.

Problem 2. The most straightforward way of solving this problem is based on calculating the partition function $Z$. For an ideal gas of $N$ atoms $Z = Z_1^N/N!$, where

$$Z_1 = \int \frac{d^3p}{(2\pi\hbar)^3} e^{-\varepsilon(|\mathbf{p}|)/\tau} = \frac{V}{(2\pi\hbar)^3} \int d^3p e^{-\varepsilon p/\tau}.$$

This gives

$$Z_1 = \frac{V}{(2\pi\hbar)^3} \times 4\pi \int_0^\infty p^2 dp e^{-\varepsilon p/\tau} = \frac{V\tau^3}{\pi^2(\hbar c)^3}.$$

We know that pressure is

$$p = -\frac{\partial F}{\partial V} = \tau \frac{\partial \log Z}{\partial V} = \frac{N\tau}{V}.$$

The internal energy is

$$U = \tau^2 \frac{\partial^2 \log Z}{\partial \tau^2} = 3N\tau.$$

Therefore we obtain $p = U/3V$.

Problem 3. From $dF = -\sigma d\tau - p dV$ we have $\sigma = -(\partial F/\partial \tau)_V$ and $p = -(\partial F/\partial V)_\tau$.

Then from $C_V = \tau (\partial^2 \sigma/\partial \tau^2)_V$ we have

$$\left( \frac{\partial C_V}{\partial V} \right)_\tau = \tau \frac{\partial^3 F}{\partial V \partial \tau^2} \equiv \tau \left( \frac{\partial^2 p}{\partial \tau^2} \right)_V.$$
For an ideal gas, the internal energy is a sum of energies of individual molecules, and therefore it is independent of $V$ for a given number of molecules. As a consequence, $\partial C_V/\partial V = 0$. On the other hand, pressure is determined only by translational motion of molecules, and therefore $p = N\tau/V$, from which we have $\partial^2 p/\partial \tau^2 = 0$, i.e. the relation between the derivatives of $C_V$ and $p$ is satisfied.

**Problem 4.** The occupation of a mode with frequency $\omega$ is $\langle s(\omega) \rangle = [\exp(\hbar \omega/\tau) - 1]^{-1}$. We will model a 2D system by a square with side $L$, as we did before, $A = L^2$. The frequency of the mode with the quantum number $\mathbf{n} = (n_x, n_y)$ is $\omega_\mathbf{n} = |\mathbf{n}|\pi c/L$. The sum over the mode quantum numbers is

$$\sum_\mathbf{n} = \frac{1}{4} \int_0^\infty 2\pi n \, dn$$

Therefore the internal energy

$$U = \sum_\mathbf{n} \hbar \omega_\mathbf{n} \langle s(\omega_\mathbf{n}) \rangle = \frac{\hbar L^2}{2\pi c^2} \int_0^\infty \omega^2 \langle s(\omega) \rangle \, d\omega = \frac{L^2 \tau^3}{2\pi \hbar c^2} \int_0^\infty \frac{x^2}{\exp(x) - 1} \, dx = \frac{A}{\pi \hbar^2 c^2 \tau^3} \zeta(3)$$

where we changed from integration over $n$ to integration over $\omega = L n/\pi c$ and then over $x = \hbar \omega/\tau$. 