Astronomy 304, STARS

Homework # 6, Due Friday, Feb. 21, 2003

Show All Your Work

1. Carroll & Ostlie: problem 10.3

   Assyming that 10 eV could be released by every atom in the Sun through chemical reactions, estimate how long the Sun could shine at its current rate through chemical processes alone. For simplicity, assume that the Sun is composed entirely of hydrogen. Is it possible that the Sun's energy is entriely chemical? Why or why not?

   The number of atoms in the Sun is

$$N = M/m_H = 1.99 \times 10^{33}/1.67 \times 10^{-24} = 1.19 \times 10^{57}$$

   The energy release by chemical reactions is

$$E = 10N\text{eV} = 1.19 \times 10^{58}\text{eV} = 1.19 \times 10^{58} \times 1.6 \times 10^{-12} = 1.91 \times 10^{46}\text{erg}$$

   This would last, at the current solar luminosity,

$$t = E/L = 1.91 \times 10^{46}/3.826 \times 10^{33} = 4.98 \times 10^{12}\text{s} = 1.6 \times 10^{5}\text{yrs} .$$

   This is much less than the age of the Earth.

2. Carroll & Ostlie: problem 10.4

   (a) What temperature would be required for two protons to collide if quantum mechanical tunneling is neglected? Assume that nuclei having velocities ten times the root-mean-square (rms) value for the Maxwell-Boltzmann disturbution can overcome the Coulomb barrier. Compare your answer with the estimated central temperature of the Sun.

$$\frac{1}{2}m(10v_{\text{rms}})^2 = 100\frac{3}{2}kT = \frac{e^2}{r}$$

   or

$$T = \frac{2}{3}10^{-2}\frac{e^2}{kr} = 0.667 \times 10^{-2}\frac{(4.8 \times 10^{-10})^2}{1.38 \times 10^{-16} \times 1. \times 10^{-13}} = 1.1 \times 10^{8}\text{K} .$$

   (b) Using Eq. (8.1) calculate the ratio of the number of protons that have velocities ten times the rms value to those moving at the rms velocity.

   The Maxwell-Boltzmann distribution of velocities (Eqn. 8.1) is

$$n(v)dv = \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-mv^2/2kT} 4\pi v^2 dv$$

   so

$$\frac{n(10\sqrt{(3kT/m)})}{n(\sqrt{(3kT/m)})} = 100e^{-100} = 3.7 \times 10^{-42}$$

(c) Assuming (incorrectly) that the Sun is pure hydrogen, estimate the number of hydrogen nuclei in the Sun. Could there be enough protons moving with a speed ten times the rms value to account for the Sun's luminosity?

$$N = M_\odot/m_{\rm H} = 1.989 \times 10^{33}/1.67 \times 10^{-24} = 1.19 \times 10^{57}$$

The number moving 10 $\times$ $\mathrm{v_{rms}}$ is

$$N(v = 10v_{\rm rms}) = 4.43 \times 10^{15}$$

Each fusion reaction 4 H $\rightarrow$ He releases 6.25 MeV/nucleon = 6.25 (MeV) $\times$ 1.6 $\times 10^{-6}$ erg/MeV = 1 $\times 10^{-5}$ erg/nucleon. The Sun needs 3.8×10^{33} ergs/s or 3.8 $\times 10^{38}$ protons consumed per second. Whether there are enough available depends on the time scale for reaching an equilibrium configuration once the initial 10^{15} nucleons moving fast enough have been fused.

3. Carroll & Ostlie: problem 10.10

Beginning with Eq. (10.56) and writing the energy generation rate in the form

$$\epsilon(T) = \epsilon'' T_8^{\alpha} \ ,$$

show that the temperture dependence for the triple alpha process, given by Eq. (10.57) is correct. ϵ'' is a function that is independent of temperature.
Hint: First take the natural logarithm of both sides of Eq. (10.56) and then differentiate with respect to $\ln T_8$. Follow the same procedure with your power law form of the equation and compare the results. You may want to make use of the relation

$$\frac{d \ln \epsilon}{d \ln T_8} = \frac{d \ln \epsilon}{\frac{1}{dT_8}} = T_8 \frac{d \ln \epsilon}{dT_8} \ .$$

Eqn. 10.56 for the energy generation rate by the triple alpha process is

$$\epsilon_{3\alpha} = 5.09 \times 10^{11} \rho^2 Y^3 T_8^{-3} f_{3\alpha} e^{-44.027 T_8^{-1}} \ \mathrm{erg/g/s}$$

where $T_8 = 10^{-8} T$. Take the logarithm of this expression,

$$\ln \epsilon_{3\alpha} = -3 \ln T_8 - 44.027/overT_8 + const.$$

Take the derivative of this with respect to $\ln T$,

$$\frac{d \ln \epsilon_{3\alpha}}{d \ln T} = n = -3 + 44.027/T_8 = 41.03$$

at $T_8 = 1$.

4. Derive the expression for the thermal conductivity of non-degenerate electrons,

$$K = \frac{3^{1/2}}{2^3\pi}\frac{k^{7/2}T^{5/2}}{m^{1/2}e^4}$$

First, one must know what the process of conduction is. Conduction is the transport of energy by the random walk of particles. Second, one must know what is meant by conductivity. Conductivity is the ratio of the conductive energy flux and the temperature gradient,

$$F = -K\frac{dT}{dz} \ .$$

Start from the general expression for the flux of anything (Q per particle) due to transport by random walk diffusion,

$$F = -\frac{1}{3}n < v > \ell\frac{dQ}{dz} \ ,$$

where ℓ is the mean free path. If the density is also varying, then it should be inside the derivative and one should deal with nQ per unit volume. The thermal energy per particle is

$$Q = E = \frac{3}{2}kT \ ,$$

the typical velocity of a non-relativistic particle is

$$< v >= \left(\frac{3kT}{m}\right)^{1/2} \ ,$$

and the mean free path is,

$$\ell = 1/n\sigma \ ,$$

so the thermal energy flux is

$$F = -\frac{3^{1/2}}{2}\sigma^{-1}k^{3/2}m^{-1/2}T^{1/2}\frac{dT}{dz} \ .$$

What is the cross-section? To determine the cross-section, one must know the process that is impeding the motion of the particles. The main impediment to the motion of electrons in a gas is Coulomb scattering by the electrostatic potentials of ions and electrons. This is a somewhat subtle problem because the electric force is a long range force and therefore many interactions at long range, which produce only a small change in direction, are as important as a few close encounters which produce a large change in direction. However, to order of magnitude, the cross-section is equal to πb^2, where b is the impact parameter at which the kinetic energy of the particle is equal to its interaction potential energy,

$$\frac{1}{2}mv^2 = e^2/b \ .$$

Thus the scattering cross-section is Thus the scattering cross-section is

$$\sigma = \pi b^2 = \pi\left(\frac{2e^2}{mv^2}\right)^2 = \pi\left(\frac{2e^2}{3kT}\right)^2 .$$

Finally, put this all together,

$$F = -\frac{3^{5/2}}{2^3\pi}\frac{k^{7/2}T^{5/2}}{m^{1/2}e^4}\frac{dT}{dz}\ . \tag{1}$$

Thus the coefficient of the gradient, which is the conductivity, is

$$K = \frac{3^{5/2}}{2^3}\frac{k^{7/2}T^{5/2}}{m^{1/2}e^4}\ . \tag{2}$$

5. (a) What is the dominant source of opacity near the surface of the Sun? (b) Why is this not the main source of opacity near the center of the Sun?

   Main source of opacity near the surface is H^-. The interior of the Sun is so hot, that collisions destroy the loosely bound H^- ion, so its abundance