Most Important Things for You to Know about Error Analysis:
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Justify your uncertainty: Give a specific reason you chose $\delta x$ as the uncertainty for the measurement of $x$. See examples in Taylor §1.5; §3.1-3.2; and §1.6, §4.1-4.6 for standard deviation and standard deviation of a mean for repeatable measurements.

Compatibility (§2.4-2.5): The whole point of quantitative measurement with uncertainties is to test hypotheses, and compare results. Say you measure $q$, and you compare it to $p$ (the expected value). Define the discrepancy as the difference of your result from the result expected by some hypothesis:
$$D = q - p = \text{measured} - \text{expected}.$$  

The best way to describe the degree of discrepancy of $p$ and $q$ is in terms of the number of standard deviations (the “t value”) of their difference from expectations:
$$t = \frac{D}{\delta D}$$
where $\delta D$ is the uncertainty of $D$ (its standard deviation, for Gaussian uncertainties).

The “two standard deviations” rule says $p$ and $q$ are compatible as long as $|t| \leq 2$.

Typically $\delta D = \sqrt{(\delta q^2 + \delta p^2)}$; or just $\delta q$ if $p$ is well known, so $\delta p$ is tiny. Best practice is to calculate $t$, then say something like “the difference is 1.6 times its uncertainty, so the measurements are compatible by the 2 standard deviation rule.” If $|t| > 2$, we would call $p$ and $q$ statistically incompatible, or call their difference statistically significant.

If your uncertainties are Gaussian, and correctly estimated, and the assumptions (hypothesis) leading to the expected value are also correct, a $|t| > 2$ deviation would occur by chance only about 5% of the time. So large $|t|$ values suggest real disagreement from what you expected, while small $|t|$ values are compatible with your hypothesis—or at least not proven to disagree. But if you measure poorly ($\delta D$ is large), your result will be compatible with most anything: not a very useful measurement.

Occasionally we use a simpler criterion compares $|D|$ with $\delta q + \delta p$ (the worst case for $\delta D$, but allowing only 1 standard deviation difference): this is just “do the error bars touch”.

Sometimes we are also interested in the fractional deviation the measured value from what we expected, which is just $D/p = (q-p)/p$; the % deviation or % difference is the same thing expressed in percent. Just because the percent difference is small, does not make it insignificant. That’s what the $t$ criterion is for. But $D/p$ is all we can report if we don’t know $\delta D$.

Know the Uncertainty Calculation Formulae (§3.3-3.7; 3.11) on inside covers of Taylor, and how/when to use them. Some hints:
The fraction uncertainties $\delta q/q$, $\delta x/x$, $\delta y/y$ all have NO UNITS (can write as a fraction, or as %, but watch the factor of 100!)

But to get $\delta q$, don’t forget to multiply $q \times (\delta q/q)$

For $q = x \pm y$, $x$, $y$, $q$, $dx$, $dy$, $dq$ all must have the same units (will want to add $q + dq$, e.g. as error bars)

How to check your calculations to see if they make sense:
$q = x + y$ always must have: $\delta q > \max (\delta x, \delta y)$
$q = x*y$ or $x/y$ always must have $\delta q/q > \max (\delta x/x, \delta y/y)$

Independent measurement: no relationship in the imperfections between the measurements; e.g. 2 students measure the same distance each with a different, but good, ruler. A measurement dominated by a systematic error (same shrunken ruler used by both students) would produce results that aren’t independent. See Chapter 4; needed to apply Chapter 3 formulas.

Random: you expect to get slightly different values each time you measure it: due to reading uncertainties, varying judgments, uncontrollable factors, or inherent properties of the measurement.

For examples, see next page.
Example of $q = \frac{x}{y}$ Uncertainty Calculation:  
$x = 10 \quad \delta x = .1 \quad y = 2.7 \quad \delta y = .2 \quad \text{so } q = 3.7$

Often easiest to do in terms of %, especially since really need uncertainties to only 1 significant figure

$$\delta q/q = \sqrt{(1\% + 8\%)} \approx 8\% \quad \text{so } \delta q \approx .08 \times q \approx .3 \quad \text{(notice 8\% → .08, the factor of 100)}$$

Whip out your calculator now: Let’s try $r = 10$ and $\delta r = .1$, so what’s the fractional error for $r$? 
$\delta r/r = 1\%$  
Now say $q = r^2$ then what’s $\delta q/q = ?$

From Eq 3.23, 3.26:

$$\delta q / q = ( |\delta q/dr| \delta r ) / r^2 = 2 \delta r / r = 2\%$$

For comparison, calculate directly (the most general way, rather than the Chapter 3 formulas, which rely on first derivative approximations):

$$(q + \delta q)/q = (r+\delta r)^2 / r^2 = 102.01/100 = 1.0201 = (q + \delta q) / q , \text{ so } \delta q/q = 2.01\% \quad \text{(same as } \delta q \rightarrow 0)$$

**A More Complicated Example Calculation** (See Step by Step: see Taylor Chapter 3.8)

$q = x^2 y + z^{1/3}$  
$x = 10 \pm .1 \quad y = 20 \pm .2 \quad z = 10000 \pm 1800$

$\delta x/x = 1\% \quad \delta y/y = 1\% \quad \delta z/z = 18\%$

let $w = z^{1/3} = 15.8 \quad x^2y =2000 \quad \text{and } q = 2015.8$

Let’s start with the product term : $x^2y$

$$\delta (x^2y) / (x^2y) = \sqrt{ \left( \frac{\delta x}{x} x^2 \right)^2 + \left( \frac{\delta y}{y} y \right)^2 } = \sqrt{ \left( 2 \times 1\% \right)^2 + \left( 1\% \right)^2 } = 2.2 \% \approx 2\%$$

notice we have used $\delta x^2 / x^2 = 2 \delta x/x$: the 2 goes inside the parentheses!

so $\delta x^2y = x^2y \times (\delta x^2y / x^2y) = 2000 \times (2\%) = 40$

Now $\delta w/w = 1/3 (\delta z/z) = 1/3 \times 18\% = 6\%$, so $\delta w = 6\% \times w \approx .9$

notice that 6% is NOT rounded up to 10%, nor is .948 rounded up to 1

in each instance we keep the first significant digit, though in the middle of a long calculation, it might make sense to keep one extra digit.

Notice also that $w$ is better known than $z$ is, and in fact has more significant digits: $15.8 \pm .9$

compared to $(10.0 \pm 1.8) \times 10^3$ !

Finally, since $q = x^2y + w$, $\delta q = \sqrt{ (40)^2 + (.9)^2 } \approx 40$

So $q = 2015.8 \pm 40$, or $2020 \pm 40 = (2.02 \pm .04) \times 10^3$ after significant figures.