Most Important Things for You to Know about Error Analysis:

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Revised October 13, 2004

Justify your uncertainty: Give a specific reason you chose δx as the uncertainty for the measurement of x. See examples in Taylor §1.5; §3.1-3.2; and §1.6, §4.1-4.6 for standard deviation and standard deviation of a mean for repeatable measurements.

Compatibility (§2.4-2.5)**: The whole point of quantitative measurement with uncertainties is to test hypotheses, and compare results.** Say you measure q, and you compare it to p (the expected value). Define the discrepancy as the difference of your result from the result expected by some hypothesis:

D = q – p = measured - expected .

The best way to describe the degree of discrepancy of p and q is in terms of the number of standard deviations (the "t value") of their difference from expectations:

***t* = D / δD** where δD is the uncertainty of D (its standard deviation, for Gaussian uncertainties).

The "**two standard deviations**" rule says p and q are compatible as long as $|t| \leq 2$ **.**

Typically $\delta D = \sqrt{(\delta q^2 + \delta p^2)}$; or just δq if p is well known, so δp is tiny. Best practice is to calculate *t*, then say something like "the difference is 1.6 times its uncertainty, so the measurements are compatible by the 2 standard deviation rule." If $|t| > 2$, we would call p and q statistically incompatible, or call their difference statistically significant.

If your uncertainties are Gaussian, and correctly estimated, and the assumptions (hypothesis) leading to the expected value are also correct, a $|t| > 2$ deviation would occur by chance only about 5% of the time. So large $|t|$ values suggest real disagreement from what you expected, while small $|t|$ values are compatible with your hypothesis—or at least not proven to disagree. But if you measure poorly (δD is large), your result will be compatible with most anything: not a very useful measurement.

Occasionally we use a simpler criterion compares $|D|$ with $\delta q + \delta p$ (the worst case for δD, but allowing only 1 standard deviation difference): this is just "do the error bars touch".

Sometimes we are also interested in the **fractional deviation** the measured value from what we expected, which is just $D/p = (q-p)/p$; the **% deviation** or **% difference** is the same thing expressed in percent. Just because the percent difference is small, does not make it insignificant. That's what the t criterion is for. But D/p is all we can report if we don't know δD.

Know the Uncertainty Calculation Formulae (§3.3-3.7; 3.11) **on inside covers of Taylor, and how/when to use them**. Some hints:

The fraction uncertainties $\delta q/q$, $\delta x/x$, $\delta y/y$ all have NO UNITS (can write as a fraction, or as %, but watch the factor of 100!)

But to get δq, **don't forget to multiply q × (δq/q)**

For $q = x \pm y$, x, y, q, dx, dy, dq all must have the same units (will want to add q + dq, e.g. as error bars)

How to check your calculations to see if they make sense:

$q = x + y$ always must have: $\delta q > \max(\delta x, \delta y)$

$q = x*y$ or x/y always must have $\delta q/q > \max(\delta x/x, \delta y/y)$

Independent measurement: no relationship in the *imperfections* between the measurements; e.g. 2 students measure the same distance each with a different, but good, ruler. A measurement dominated by a systematic error (same shrunken ruler used by both students) would produce results that aren't independent. See Chapter 4; needed to apply Chapter 3 formulas.

Random: you expect to get slightly different values each time you measure it: due to reading uncertainties, varying judgments, uncontrollable factors, or inherent properties of the measurement.

For examples, see next page.

Example of $q = x/y$ Uncertainty Calculation: $x = 10$ $\delta x = .1$ $y = 2.7$ $\delta y = .2$ so $q = 3.7$

Often easiest to do in terms of %, especially since really *need uncertainties to only 1 significant figure*

$\delta q/q = \sqrt{(1\% + 8\%)} \approx 8\%$ so $\delta q \approx .08 \times q \approx .3$ (notice 8% → .08, the factor of 100)

Whip out your calculator now: Let's try $r=10$ and $\delta r = .1$, so what's the fractional error for r?

$\delta r/r = 1\%$ Now say $q= r^2$ then what's $\delta q/q = ?$

From Eq 3.23, 3.26:

$\delta q / q = (|dq/dr|\ \delta r) / r^2 = 2\ \delta r / r = 2\%$

For comparison, calculate directly (the most general way, rather than the Chapter 3 formulas, which rely on first derivative approximations):

$(q + \delta q)/q = (r+\delta r)^2 / r^2 = 102.01/100 = 1.0201 = (q + \delta q) / q$, so $\delta q/q = 2.01\%$ (same as $\delta q \to 0$)

A More Complicated Example Calculation (See Step by Step: see Taylor Chapter 3.8)

$q = x^2 y + z^{1/3}$ $x = 10 \pm .1$ $y = 20 \pm .2$ $z = 10000 \pm 1800$

$\delta x/x = 1\%$ $\delta y/y = 1\%$ $\delta z/z = 18\%$

let $w = z^{1/3} = 15.8$ $x^2y =2000$ and $q = 2015.8$

Let's start with the product term : x^2y

$\delta (x^2y) / (x^2y) = \sqrt{\{ (\delta x^2/x^2)^2 + (\delta y/y)^2 \}} = \sqrt{\{ (2 \times 1\%)^2 + (1\%)^2 \}} = 2.2\ \% \approx 2\%$

notice we have used $\delta x^2 / x^2 = 2\ \delta x/x$: **the 2 goes inside the parentheses!**

so $\delta x^2y = x^2y \times (\delta x^2y / x^2y) = 2000 \times (2\%) = 40$

Now $\delta w/w = 1/3\ (\delta z/z) = 1/3 \times 18\%= 6\%$, so $\delta w = 6\% \times w \approx .9$

notice that 6% is NOT rounded up to 10%, nor is .948 rounded up to 1

in each instance we keep the first significant digit, though in the middle of a long calculation, it might make sense to keep one extra digit.

Notice also that w is better known than z is, and in fact has *more* significant digits: $15.8 \pm .9$ compared to $(10.0 \pm 1.8) \times 10^3$!

Finally, since $q = x^2y + w$, $\delta q= \sqrt{\{ (40)^2 + (.9)^2 \}} \approx 40$

So $q = 2015.8 \pm 40$, or $2020 \pm 40 = (2.02 \pm .04) \times 10^3$ after significant figures.