# Experiment 3

# The Simple Pendulum

## Suggested Reading for this lab

Read Taylor chapter 5. (You can skip section 5.6.IV if you aren't comfortable with partial derivatives; for a simpler look at the material of section 5.7 see below).
Halliday, Resnick, Walker Ch 8, 14 as needed (pendulum, and definitions of period, amplitude)

**Taylor Section 5.7 without partial derivatives:**
We want to know the uncertainty of the mean value, $m = (x_1 + x_2 + \ldots + x_N)/N$ . Now N is a constant, so I can start by determining the uncertainty of $S = N\,m = (x_1 + x_2 + \ldots + x_N)$. But by the arguments of 5.6.III (and eq 3.16), the uncertainty of S is just given by:

$\sigma_S = \sqrt{\{ \sigma_x^2 + \sigma_x^2 + \ldots + \sigma_x^2 \}} = \sqrt{\{ N\sigma_x^2 \}} = \sigma_x \sqrt{N}$

since the standard deviation of each of the $x_i$ is just $\sigma_x$

But $m = S/N$, and N is a constant, so $\sigma_m = \sigma_S / N$, which gives our final result

$\sigma_m = ( \sigma_x \sqrt{N} ) / N$ or $\boldsymbol{\sigma_m = \sigma_x / \sqrt{N}}$

### Homework 6: turn in as part of your preparation for this the first week of this experiment.

Do problems 5.17 and 5.18. Don't just write down the answers, but explain how you got them.

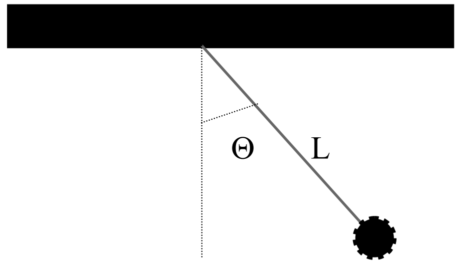
### Homework 7: Turn in at start of 2nd week of experiment.

Do problems 5.34 and 5.35 Don't just write down the answers, but explain how you got them.
For a given initial angle $\Theta$, calculate the $h$, the maximum height of the bob above equilibrium used in Eqs 15-16.
Extra Credit: Verify by direct substitution that Eq. 7 is a solution of Eq. 6.

## Goals

1. Measure g with the simple pendulum
2. Improve measurement accuracy by averaging
3. Study the amplitude and mass dependence of the period of a pendulum
4. Study energy conservation
5. Examine the propagation of error in derived physical quantities

# 1. Theoretical Introduction




### 1.1 The period of the pendulum

The simple pendulum, shown above, consists of a mass m (the "bob") suspended from a pivot by a massless string. We will idealize the bob as a point mass located at the center of mass of the bob. The distance from the point of pivot to the center of mass of the ball is designated by L in figure. When the ball is displaced from its resting positions the string makes an angle $\Theta$ with the vertical. The component of the gravitational force in the tangential direction acts to restore it to its equilibrium position. Calling *x* the tangential coordinate ($x = L\Theta$, the arc length, with $x=0$ at $\Theta=0$), the restoring force is:

$$F_x = -mg\sin\Theta \qquad (1)$$

The tension of the string T, in the direction toward the point of suspension, is equal in magnitude and opposite in direction to the component of the gravitational force acting in that direction. The mass is accelerated only in tangential direction perpendicular to the string. Using Newton's Second Law, F= ma, the relation between the tangential displacement x and the corresponding change in angle $\Theta$ is given by:

$$F_x = ma_x = m\frac{d^2x}{dt^2} = m\frac{d^2}{dt^2}(L\Theta) = mL\,\frac{d^2\Theta}{dt^2} \qquad (2)$$

Inserting the expression for the restoring force, our equation of motion becomes:

$$mL\frac{d^2\Theta}{dt^2} = -mg\sin\Theta \qquad (3)$$

or

$$\frac{d^2\Theta}{dt^2} + \frac{g}{L}\sin\Theta = 0 \qquad (4)$$

This turns out to be hard to solve, but we can simplify it by using the fact that for small angles $\Theta$, we may expand $\sin\Theta$ as follows :

$$\sin\Theta \approx \Theta - \frac{\Theta^3}{6} + \frac{\Theta^5}{120} \qquad (5)$$

Then, if we assume $\Theta$ to be small and keep only the first term, Eq 4 becomes:

$$\frac{d^2\Theta}{dt^2} + \frac{g}{L}\Theta = 0 \qquad (6)$$

This is much easier to solve analytically, and the solution to differential equation (6) is:

$$\Theta = \Theta_o \sin(t\sqrt{\frac{g}{L}}) \tag{7}$$

Because the sine function repeats itself whenever its argument changes by 2π, T, the time for one period, can be found by:

$$T\sqrt{\frac{g}{L}} = 2\pi \tag{8}$$

or

$$T = 2\pi\sqrt{\frac{L}{g}} \tag{9}$$

Now we can solve for g in terms of two quantities we can measure. Or, we can find a relationship between measured quantities such that g is related to the slope of a plot when we vary the values:

$$g = 4\pi^2 \frac{L}{T^2} \quad or \quad L = 4\pi^2 g T^2 \tag{10}$$

Thus as long as the small angle approximation of Eq. 5 is valid, the period is independent of the amplitude, $\Theta_o$ and mass, m. Measurement of the period T , and the length L , permit a determination of the gravitational constant, g. If $\Theta_o$ is not small enough, Eq. 6 will not be valid and the period will depend on $\Theta_o$ and will actually increase with the amplitude (see Appendix B).

## 1.2 Energy analysis of the pendulum

For a pendulum swinging back and forth, the mechanical energy, E, shifts between kinetic and potential energy, but remains constant:

$$E = K + U \tag{11}$$

$$U = mgy \tag{12}$$

$$K = \frac{1}{2}mv^2 \tag{13}$$

Here $y$ is vertical displacement from equilibrium, and $v$ is velocity of the bob. When the bob is at the maximum amplitude, $x = x_m$, and y = h (the maximum vertical displacement). At this point, $v = 0$: there is no kinetic energy, so all the energy is potential energy. The bob has greatest speed at its lowest point, hence all the energy is kinetic, and $U = 0$.

Conservation of mechanical energy for these two instants can be expressed as:

$$K_0 + U_0 = K_m + U_m \tag{14}$$

where subscript *0* stands for values evaluated at the equilibrium position (x = 0, y=0) and the subscript $m$ stands for values at highest point of the oscillation ($x = x_{max}$, y=h). Then we can evaluate each term and find

$$\frac{1}{2}mv_0^2 + 0 = 0 + mgh \tag{15}$$

This equation relates the maximum velocity (at x=0) to the maximum height and the value of g. Curiously, the maximum velocity is achieved at the equilibrium position!

We can solve (15) for $v_0^{\ 2}$ and obtain:

$$v_o^{\ 2} = \left(\frac{\Delta x}{\Delta t}\right)^2 = 2gh \tag{16}$$

This expression will be useful when we study energy conservation.

# Experimental Procedure

## 2. Preliminaries

A bob is suspended from a pivot by a string. A protractor is placed below the pivot which allows us to set the pendulum oscillation at different angles (amplitude). We can measure the period of the pendulum using:

1. Manual timing with a digital clock.
2. Automatic timing with a photogate timer.

You should then be able to compute g from the period of the pendulum and the length of the string.

Measure the length L between the pivot of the pendulum and the center of mass of the bob as accurately as possible. You may need several measurement, or measurement strategies, to find L. Assess the role of statistical and systematic uncertainties in your value for L.

### Questions for preliminary discussion

2.1 Draw a diagram of forces acting on the bob.
2.2 What component of force causes oscillation?
2.3 Should the period of the pendulum depend on the mass?
2.4 Should the period of the pendulum depend on the amplitude?
2.5 What is the criterion for a small-amplitude oscillation?
2.6 Would the measurements be most accurate with a long or a short string?
2.7 Discuss sources of systematic and random errors in this experiment.
2.8 Discuss what tables you will need to organize your measurements and the uncertainty calculations.

## 3. Manual measurements of T:

It is most accurate to begin timing the swing of the pendulum at its lowest point because then the ball moves most quickly and takes the least time to pass by. The amplitude of the swing should be large in order to maximize the speed of the pendulum at that point. On the other hand, $\Theta$ must be kept small enough that the approximation $\sin\Theta \approx \Theta$ remains valid. As a compromise, take the initial amplitude to be about 0.1 radians (~$6^o$).

1. Using timer, measure 25 complete cycles and calculate the period. Start and stop the measurements at the lowest point of the swing. Repeat the period measurement **ten times**.
2. From these ten measurements of the period, calculate the mean period and the standard deviation of the mean period using Kgraph or Excel. Do not round off your numbers too early in your calculations lest you lose accuracy in your final result. Using Eq. 10, calculate *g* and its uncertainty.

## 4. Automatic measurement of T (for various masses and lengths)

Set the photogate on PEND position. Practice timing the period with using the photogate a few times. Figure out how the gate works by moving the bob through the gate *slowly, by hand*. How many periods does the photo-gate measure? (Write it in your lab book!)

1. From three measurements of the period with the photo-gate, again calculate the mean, and, the standard deviation of the mean. Calculate g and its uncertainty.

2. Change the bob and find the period for other masses. Are your results consistent with Eq 10?

3. Now measure the period with the length of the string reduced to L/2, L/3, L/4, and L/5. Make a plot of $T^2$ vs. $L$. Figure out how g is related to the slope of this plot and find and $g$ and its uncertainty from this method. (For Kgraph to calculate uncertainties for you, you'll need error estimates for L).

4. An oscillating solid rod with uniform cross section also forms a pendulum. If the rod is suspended from one end, its period is given by $T = 2\pi\sqrt{\dfrac{2L}{3g}}$ . (This result is derived in Appendix A.) Find the period of the rod as in (4.1).

## 5. Amplitude dependence of the period

If the amplitude of oscillation of a pendulum is not sufficiently small, its period will depend on amplitude. Thus Eq. 9 will not be valid. See Appendix B for a brief discussion.

For this part, remove the rod and go back to the string and bob. Hint: the largest bob may not be appropriate for this measurement. Why? Use the photogate timer to measure the period of the pendulum for a series of starting angles. Begin with 30° (about 0.5 radians) maximum and repeat for approximately 25°, 20°, 15°, 10°, and 6°. Now we have data to test Eq. B3, which has the form

$$T = T(\Theta) = To\ [1 + A\ \Theta^2 + \text{higher terms}],$$

where To is the small amplitude period defined in Eq. 9, and A = 1/16 (provided the angle Θ has been converted to radians). So a nice way of testing the equation is to calculate the ratio $T(\Theta)$ / To as a function of $\Theta^2$, where $T(\Theta)$ is the period you measured with a given initial angle. If Eq B3 holds, and the higher terms are negligible, the plot should be close to linear. Perform a linear least squares fit of the data to T(Θ)/To vs. $\Theta^2$ . Compare your slope with the theoretical value of 1/16. It is not necessary to find the uncertainty of the slope: just say by how many percent your coefficient differs from the expected one, and whether the data follow the expected trend.

## 6. Conservation of Energy

Now we will test the idea of conservation of energy by measuring the velocity of the bob (kinetic energy) as a function of its release height (potential energy). Measure the diameter of the bob with maximum accuracy. Set the photogate in *Gate* position. Determine the vertical displacement from equilibrium h for the angles Θ = 30° to 5° in 5° intervals. When the bob passes the equilibrium point, the photogate timer measures the time interval over which the bob interrupts the light. From the time intervals, find $v_0$ and plot $v_0^2$ vs. h (Eq. 16). Perform a linear least-squares fit to the data to obtain g. It is not necessary to do an extended uncertainty analysis here, either—just discuss whether the data follows the expected trend and calculate the value of g you obtain from the slope.

### 7. Damped Pendulum

Next we consider the damped pendulum. In any real pendulum, frictional losses decrease the energy of the pendulum as time goes on. We now concentrate on this aspect of the pendulum by measuring the peak velocity of the pendulum as its swings slow down. This will allow you to find the functional form of the energy decay.

Choose a string length of about 20 cm and set the photogate timer switches to GATE with MEM *on*. Take 10 successive measurements of the bob velocity, at 15 s intervals. At each interval, determine the velocity by reading the manual timer. Your final kinetic energy should be < 5% if the initial value. If not, choose a different interval for your measurements.

Calculate *E* (recall from Eq 15 that the potential energy is 0 at the bottom of the swing) and plot the kinetic energy vs. t. Explain the graph in your report. **Extra credit:** Can you transform your plot variables so that a straight line results? Can you define a characteristic time for the energy decay?

### 8. Phase Space Portrait (Extra Credit)

When the bob is released from an angle $\Theta$ , the pendulum oscillates between $+\Theta$ and $-\Theta$ . In one cycle, the bob passes each point twice. The momentum *p* at these two passages is first positive, and then negative. Phase space considers motion with both position and momentum as coordinates, so it considers these two passages as two distinct points ($\theta$, p) and ($\theta$, -p); in phase space, the cycle is not complete until the object returns to the same phase space point.

Make a *phase space* plot describing the motion of a pendulum. Your phase space plot should have orthogonal axes for *p* and $\Theta$ . For example, assume that $\Theta = 30°$ is your initial angle. Then when $\Theta = 30°$, $U=mgh$ and $p = 0$ . When $\Theta = 0°$, $p= -mv$ (left direction is negative; we've just assumed a symbolic value for *p*). At $\Theta = -30°$, $p = 0$ again; and on returning to $\Theta = 0°$ , p = +mv (going right). Connect these four points with smooth line, forming an ellipse. Repeat for smaller initial angles. Now, imagine that the pendulum's amplitude is continuously decreasing to zero. Sketch the diagram for this situation. For a pendulum that is being damped, the diagram encompasses its complete dynamics, from start to finish. Explain this in your report.

## 9. Analysis of Results

Make a summary table listing all your methods of measuring g, the values and (if applicable), the uncertainty, the percent difference from the accepted value of g (9.804 $m/s^2$ ), and where possible, the t value of the discrepancy between your measured value and the expected one.

Address the following questions in your report:

9.1. What is the point of measuring 250 cycles (25 x 10) in Part A? Would it have been as accurate to measure one cycle 250 separate times?

9.2. Which value of g is more accurate, the one obtained by hand-timing or obtained with the photogate? Which method is faster to use?

9.3. Assuming that your uncertainties are random, how many hand-timing measurements should be done to make the two sets of measurements equally precise?

9.4. Which quantity, T or L, makes the larger contribution to the fractional uncertainty in g? Does this suggest a way to improve the experiment?

9.5. Compare your most accurate value of g with 9.804 $m/s^2$. Do you have a significant discrepancy? Discuss this quantitatively from a statistical point of view.

9.6. What the major source of error in your experimental verification of $v_0^2 = 2gh$?

9.7. For which parts of the experiment did your results accord with your expectations? Which did not? Why?

# Appendix A. The Solid Rod Pendulum

For a physical pendulum like a rod pivoting at one end, a restoring torque $\tau$ reduces the angle when the pendulum is displaced from its equilibrium,

$$\tau = (mg\sin\Theta)\,(d)\,, \qquad \text{(A1)}$$

where $mg\sin\Theta$ is the tangential component of the force and d is the distance from the pivot to the center of mass. For small angles Eq. A1 becomes:

$$\tau \approx -(mgd)\Theta \qquad \text{(A2)}$$

which is the angular form of the Hooke's law. Writing the differential form of the torque and setting it equal to Eq. B2, we obtain the equation of motion:

$$\tau = rF = rm\frac{d^2(r\Theta}{dt^2} = mr^2\frac{d^2\Theta}{dt^2} = -(mgd)\Theta \qquad \text{(A3)}$$

or

$$\frac{d^2\Theta}{dt^2} = -(\frac{mgd}{\mathrm{I}})\Theta \qquad \text{(A4)}$$

where $\mathrm{I} = \sum mr^2$ is the moment of inertia. The solution to (A4) is:

$$\Theta = \Theta_o \sin\left(t\sqrt{\frac{mgd}{\mathrm{I}}}\right) \qquad \text{(A5)}$$

so that the time for one period is

$$\mathrm{T} = 2\pi\sqrt{\frac{I}{mgd}}\,. \qquad \text{(A6)}$$

For the rod $\mathrm{I} = \frac{1}{3}mL^2$ and the center of mass distance d=L/2 where *L* is the length and *m* is the mass of the rod. Therefore, we find that

$$\mathrm{T} = 2\pi\sqrt{\frac{2L}{3g}} \qquad \text{(A7)}$$

# Appendix B. The Finite Amplitude Pendulum

We wish to solve Eq.4 exactly:

$$\frac{d^2\Theta}{dt^2} + \frac{g}{L}\sin\Theta = 0 \tag{B1}$$

We may formally write the solution for the period, T, as an integral over the angle $\Theta$ (in radians):

$$T = 2\sqrt{\frac{L}{g}}\int_0^{\Theta_o}\left[\left(\sin\frac{\Theta}{2}\right)^2 - \left(\sin\frac{\Theta_o}{2}\right)^2\right]^{-\frac{1}{2}} d\Theta \tag{B2}$$

Integrals of this form belong to a class of elliptic integrals which do not have closed form solutions. If the angle $\Theta_o$ is sufficiently small, solutions to any desired degree of accuracy can be obtained by doing series expansions in the angles. We state the result without proof below:

$$T = 2\pi\sqrt{\frac{L}{g}}\left(1 + \frac{1}{16}\Theta_o^2 + \frac{11}{3072}\Theta_o^4 + \cdots\right) \tag{B3}$$

Note that the period *increases* as the amplitude *increases*.